TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 62 NUMBER 8

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OCTOBER 1955

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THIS IS THE OFFICIAL JOURNAL OF THE

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year. To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

NETS AND FILTERS IN TOPOLOGY

R. G. BARTLE, Yale University

Since the notion of a limit is of utmost importance throughout all of analysis, it is only proper that it has received frequent and varied treatment at the hands of many mathematicians. Of the several theories* available, there are two which appear to be most popular in current use. One is that of *net*, or *directed system*, which was initiated by E. H. Moore and H. L. Smith [7]† and has been discussed and improved by J. L. Kelley [4]; the other is that of *filter* and was originated by H. Cartan [2] [3]. It appears that the former notion is predominant in this country, while the filter theory reigns supreme in France.

It is well-known that these theories are equivalent in the sense that there are no proofs attainable by one that cannot be reached by the other. However, it is undeniable that each of these approaches has advantages—however psychological—not immediately possessed by the other. The use of nets parallels very closely standard constructions involving sequences, and has the pronounced advantage that sequential arguments may quite readily be adapted to them. Nets also render service in defining integrals as limits over the partitions of the interval. For these reasons they are very natural for many "analytical" arguments. On the other hand, the fact that filters are dual ideals in the algebra of sets renders a certain symmetry to their use, and often an "algebraic elegance." In addition, they are admirably suited for certain transfinite arguments, and enjoy a uniqueness not possessed by nets.

It would appear, therefore, that to neglect one notion, or to put undue emphasis on the other, is a shackle which the student assumes at his own peril. The equivalence of these concepts noted before assures that an argument involving filters can always be translated into one involving nets, and *vice versa*. Yet no matter how simple or how routine such a translation may be, it is likely to be shunned by most of us.

It is the purpose of this note to call explicit attention to certain simple relations between filters and nets and to construct each out of the other so that such a translation procedure is not necessary and so that these notions may be used simultaneously. It is felt that in this way the advantages of each will be preserved.

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* We give some references at the end of the paper; for others, see McShane [6]. *Added in proof.* Since this paper was written, a textbook on topology by J. L. Kelley, making systematic use of nets, has appeared. Mention should also be made of a paper by G. Bruns and J. Schmidt [9] which discusses both nets and filters and gives many references. Finally, it has been pointed out (see Math. Reviews, vol. 13, 1952, p. 829) that the notion of a subnet was employed by E. H. Moore long before the publication of [4].

† Numbers in brackets refer to the references at the end.

1. Preliminary definitions. We will assume a casual acquaintance, though not a detailed familiarity, with these theories. We refer the reader to the notes of Kelley [4] and McShane [5] on nets, and the discussion in Bourbaki [1, §§5, 6] concerning filters. However, partly for the convenience of the reader and partly to establish our notation we now give the basic definitions.

A *directed set* $A=\{\alpha\}$ is a set of elements of any kind in which there is an order relation defined between certain pairs of elements, written "$\leqq$", and satisfying the conditions

(i) $\alpha\leqq\alpha$, for all $\alpha\in A$;
(ii) if $\alpha_1\leqq\alpha_2$ and $\alpha_2\leqq\alpha_3$, then $\alpha_1\leqq\alpha_3$;
(iii) for any pair α_1, α_2 in A there is an $\alpha_3\in A$ such that $\alpha_1\leqq\alpha_3$ and $\alpha_2\leqq\alpha_3$.

A *net*, or a *directed system*, in an abstract set X is a function on a directed set with values in X.

In remembrance of sequences it is common to denote a net $\mathfrak{x}$ on A to X by the notation $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$. Some authors, however, adhere to the functional notation.

If E is a subset of X, we say that the net $\mathfrak{x}$ is *ultimately in* E if there is some index α_0 (depending on E) such that if $\alpha\geqq\alpha_0$ then $x_\alpha\in E$. If X is a topological space, we say that the net $\mathfrak{x}$ *converges to an element* $x_0\in X$ in case that $\mathfrak{x}$ is ultimately in every neighborhood of x_0.

This much is all very standard and natural. Less standard, but of fundamental importance in many applications is the notion of a subnet due to Kelley [4]. (The unacquainted reader will want to refer to McShane [5] for an excellent motivation and discussion of this idea.) We suppose that we are in possession of a net $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$. In this case, a net $\mathfrak{y}=\{y_\beta\}_{\beta\in B}$ is said to be a *subnet* of $\mathfrak{x}$ in case there is a mapping $\pi: B\to A$ with the properties*

(i) $y_\beta=x_{\pi(\beta)}$, for all $\beta\in B$;
(ii) given any $\alpha_0\in A$, there is a $\beta_0\in B$ such that if $\beta\geqq\beta_0$ then $\pi(\beta)\geqq\alpha_0$.

We emphasize that the domains A and B of the nets $\mathfrak{x}$ and $\mathfrak{y}$ may be totally unrelated. What is essential is that the range of $\mathfrak{y}$ is contained in the range of $\mathfrak{x}$ (condition (i)) and that the net $\mathfrak{y}$ "goes as far as" the net $\mathfrak{x}$ in the sense made precise in condition (ii).

As a simple example of a special case of a subnet, we mention the following. Let α_0 be an arbitrary index; then if $A_0=\{\alpha\in A:\alpha\geqq\alpha_0\}$ is allowed to inherit its ordering as a subset of A, it remains a directed set and the net $\{x_\alpha\}_{\alpha\in A_0}$ is clearly a subnet of $\mathfrak{x}$. Subnets formed in this manner are sometimes called *residual subnets* [8, p. 10]; they are very useful. Unfortunately, however, neither these residual subnets or the related *cofinal subnets* (see [8]) are sufficient for all purposes.

We now review the necessary definitions in the filter theory. Surprisingly

* The reader will observe that the same symbol is being used to indicate the order in both A and B.

enough, at least as regards convergence, the most important concept in this circle of ideas is that of the filter base rather than the filter itself. A non-void collection $\mathfrak{B}$ of non-void subsets of an abstract set X is called a *filter base* in X, provided that the intersection of two sets in $\mathfrak{B}$ contains a set in $\mathfrak{B}$. A *filter* in the set X is a non-void collection $\mathfrak{F}$ of subsets of X with the properties

(i) every set which contains a set in $\mathfrak{F}$ is itself in $\mathfrak{F}$;
(ii) every finite intersection of sets in $\mathfrak{F}$ is in $\mathfrak{F}$;
(iii) the void set $\varnothing$ does not belong to $\mathfrak{F}$.

Clearly a filter is itself a filter base; conversely, if $\mathfrak{B}$ is a filter base in X, then the collection of all subsets of X which contain a set in $\mathfrak{B}$ is seen to be a filter—called the *filter generated by* $\mathfrak{B}$.

We say that the filter base $\mathfrak{B}$ is *ultimately in* a subset E of X if E contains some set from $\mathfrak{B}$. Similarly, if X is a topological space, the filter base $\mathfrak{B}$ *converges to an element* $x_0 \in X$ if it is ultimately in every neighborhood of x_0.

If $\mathfrak{D}$ and $\mathfrak{B}$ are two filter bases, we say that $\mathfrak{D}$ *is a refinement of* $\mathfrak{B}$ if every set in $\mathfrak{B}$ contains some set in $\mathfrak{D}$. If this is true, we write $\mathfrak{D} \supseteq \mathfrak{B}$. It is readily seen that if $\mathfrak{D} \supseteq \mathfrak{B}$ and if $\mathfrak{G}$ and $\mathfrak{F}$ are the filters they generate, then $\mathfrak{G} \supseteq \mathfrak{F}$ both in the sense we have just defined for filter bases and in the sense that every set in $\mathfrak{F}$ is also in $\mathfrak{G}$.

If $\mathfrak{B}$ is a filter base and $E_0 \in \mathfrak{B}$, then it is evident that the set $\mathfrak{B}_0 = \{E \cap E_0 : E \in \mathfrak{B}\}$ is also a filter base and that $\mathfrak{B}_0 \supseteq \mathfrak{B}$. (This process corresponds exactly to taking a residual subnet of a given net.) A somewhat less trivial construction of a refining filter base is indicated in the following proposition.

1.1. PROPOSITION. *Let $\mathfrak{B} = \{E_\lambda\}$ be a filter base in a set X, and let E be any subset of X. Then at least one of the collections $\{E \cap E_\lambda\}$ and $\{E' \cap E_\lambda\}$ forms a filter base in X which is a refinement of $\mathfrak{B}$.*

Proof. If $E = X$, or if $E = \varnothing$, the statement is trivial, so that we may suppose that E and its complement E' are non-void. If each set $E \cap E_\lambda$ is non-void, the statement follows readily. If there exists a λ_0 such that $E \cap E_{\lambda_0} = \varnothing$, then $E_{\lambda_0} \subseteq E'$. Hence, for any λ, we have $E' \cap E_\lambda \supseteq E_{\lambda_0} \cap E_\lambda \neq \varnothing$ so that the sets $\{E' \cap E_\lambda\}$ are non-void and form a filter base.

It may happen that both of these collections are filter bases that refine $\mathfrak{B}$. The important fact is that we can continue to add more and more sets to $\mathfrak{B}$, getting finer and finer filter bases.

2. Relations between nets and filters. We are now sufficiently well-stocked with definitions to make some observations. While most of these are very simple remarks, we state them formally as propositions.

2.1. PROPOSITION. (a) *If $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$ is a net in an abstract set X, and if $E(\alpha) = \{x_\lambda : \lambda \geqq \alpha\}$, then the collection $\mathfrak{B}(\mathfrak{x}) = \{E(\alpha)\}$ is a filter base in X, called* ***the filter base associated with the net*** *$\mathfrak{x}$.*

(b) *If the net $\mathfrak{x}$ is ultimately in some set E, then $\mathfrak{B}(\mathfrak{x})$ is ultimately in E.*

(c) *If* $\mathfrak{y}=\{y_\beta\}_{\beta\in B}$ *is a subnet of* $\mathfrak{x}$ *and if* $\mathfrak{B}(\mathfrak{y})$ *is the filter base associated with* $\mathfrak{y}$, *then* $\mathfrak{B}(\mathfrak{y})$ *is a refinement of* $\mathfrak{B}(\mathfrak{x})$.

Proof. Let $E(\alpha_1)$ and $E(\alpha_2)$ be arbitrary sets in $\mathfrak{B}(\mathfrak{x})$. Since $A=\{\alpha\}$ is a directed set, there is an α_3 with $\alpha_1\leqq\alpha_3$ and $\alpha_2\leqq\alpha_3$. Clearly $E(\alpha_3)\subseteq E(\alpha_1)\cap E(\alpha_2)$ and (a) is proved. To prove (b), we note that there exists an α_0 such that if $\alpha\geqq\alpha_0$ then $x_\alpha\in E$. Consequently $E(\alpha_0)\subseteq E$, and $\mathfrak{B}(\mathfrak{x})$ is ultimately in E. Finally, let $E(\alpha_0)\in\mathfrak{B}(\mathfrak{x})$. By condition (ii) in the definition of subnet, there exists a β_0 such that if $\beta\geqq\beta_0$, then $\pi(\beta)\geqq\alpha_0$. Since $F(\beta_0)=\{y_\beta:\beta\geqq\beta_0\}=\{x_{\pi(\beta)}:\beta\geqq\beta_0\}$, we conclude that $F(\beta_0)\subseteq E(\alpha_0)$. This shows that $\mathfrak{B}(\mathfrak{y})$ is a refinement of $\mathfrak{B}(\mathfrak{x})$ and (c) is proved.

2.2. COROLLARY. *If* X *is a topological space and* $\mathfrak{x}$ *is a net in* X *which converges to* $x_0\in X$, *then the filter base* $\mathfrak{B}(\mathfrak{x})$ *associated with* $\mathfrak{x}$ *converges to* x_0.

Proof. This follows immediately from 2.1(b).

We now show that filter bases give rise in a natural manner to nets.*

2.3. PROPOSITION. (a) *Let* $\mathfrak{B}=\{E_\alpha\}_{\alpha\in A}$ *be a filter base in an abstract set* X, *where* A *is some index class. Order* $A=\{\alpha\}$ *by the requirement that* $\alpha_1\leqq\alpha_2$ *means* $E_{\alpha_1}\supseteq E_{\alpha_2}$. *With this ordering* A *is a directed set, and if from each set* $E_\alpha\in\mathfrak{B}$ *we pick an arbitrary point* $x_\alpha\in E_\alpha$, *then* $\mathfrak{x}(\mathfrak{B})=\{x_\alpha\}_{\alpha\in A}$ *is a net in* X. *Any net* $\mathfrak{x}(\mathfrak{B})$ *obtained in this manner is called* **a net associated with the filter base** $\mathfrak{B}$.

(b) *If the filter base* $\mathfrak{B}$ *is ultimately in some subset* E *of* X, *then any net* $\mathfrak{x}(\mathfrak{B})$ *associated with* $\mathfrak{B}$ *is ultimately in* E.

Proof. The fact that A is directed under the indicated ordering is a consequence of the fact that the intersection of any two sets in $\mathfrak{B}$ contains another set in $\mathfrak{B}$. To prove (b), let $E_{\alpha_0}\subseteq E$; then if $\alpha\geqq\alpha_0$ we have $x_\alpha\in E_\alpha\subseteq E_{\alpha_0}$ and so $x_\alpha\in E$ for $\alpha\geqq\alpha_0$.

2.4. COROLLARY. *If* X *is a topological space and* $\mathfrak{B}$ *a filter base in* X *which converges to* $x_0\in X$, *then any net* $\mathfrak{x}(\mathfrak{B})$ *associated with* $\mathfrak{B}$ *converges to* x_0.

Proof. This follows immediately from 2.3(b).

The reader will have observed that Proposition 2.3 lacks a part asserting that a filter base which refines $\mathfrak{B}$ gives rise to a subnet of a prescribed associated $\mathfrak{x}(\mathfrak{B})$. This is not always the case, as the reader may readily show. However, if we start with a net $\mathfrak{x}$, and form $\mathfrak{B}(\mathfrak{x})$, then any refinement $\mathfrak{D}$ of $\mathfrak{B}(\mathfrak{x})$ generates a subnet of $\mathfrak{x}$ in a manner similar to that prescribed in 2.3. This statement will now be proved.

* The referee has pointed out that one can also construct nets from filters in the following manner: if $\mathfrak{F}$ is a filter, let A be the set of all pairs (x, F), where $x\in F\in\mathfrak{F}$. If A is ordered by the requirement $(x, F)\leqq(x', F')$ if and only if $F\supseteq F'$, then the mapping $(x, F)\to x$ gives a net. Further, the set F is the image under this mapping of all the sets (x', F') for which $(x, F)\leqq(x', F')$. The reader will observe the similarity between this comment and the constructions used in Propositions 2.1 and 2.3.

2.5. PROPOSITION. *Let $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$ be a net in X and let $\mathfrak{B}(\mathfrak{x})$ be the associated filter base. Let $\mathfrak{D}=\{F_\beta\}_{\beta\in B}$ be a refinement of $\mathfrak{B}(\mathfrak{x})$, and order $B=\{\beta\}$ as in* 2.3(a). *Using the points x_α, rename as y_β in any manner subject only to the restriction that x_α cannot be renamed y_β unless it belongs to F_β. Then $\mathfrak{y}=\{y_\beta\}_{\beta\in B}$ is a subnet of $\mathfrak{x}$.*

Proof. It is not *a priori* evident that a given set in $\mathfrak{D}$ need contain any point x_α. Assume that F_β does not, and take a set $E_{\alpha'}\in\mathfrak{B}(\mathfrak{x})$. Since $\mathfrak{D}$ is a refinement of $\mathfrak{B}(\mathfrak{x})$, $E_{\alpha'}$ contains some set $F_{\beta'}\in\mathfrak{D}$ and $F_\beta\cap F_{\beta'}=\varnothing$, which is a contradiction. Thus the indicated construction can be carried out and it is clear that $\mathfrak{y}$ is a net in X. It remains to show that it is a subnet of $\mathfrak{x}$. Let $\pi:B\to A$ be the function that renames the x_α; evidently π satisfies condition (i) in the definition of subnet. To see that it satisfies (ii) we observe that for each $E(\alpha_0)\in\mathfrak{B}(\mathfrak{x})$ there is a β_0 such that if $\beta\geqq\beta_0$, then $F_\beta\subseteq E(\alpha_0)$. This is true since $\mathfrak{D}$ is a refinement of $\mathfrak{B}(\mathfrak{x})$. Since $x_{\pi(\beta)}=y_\beta\in F_\beta\subseteq E(\alpha_0)$, we conclude that $\pi(\beta)\geqq\alpha_0$ so that $\mathfrak{y}$ is a subnet of $\mathfrak{x}$.

We would now like to indicate briefly how the two theories may be utilized at the same time with advantage.

2.6. *Example.* We say that a point x_0 in a topological space X is a *cluster point*, or an *adherence point*, of the filter base $\mathfrak{B}$ if every neighborhood of x_0 has a non-void intersection with every set in $\mathfrak{B}$; x_0 is a cluster point of the net $\mathfrak{x}$ if it is a cluster point of the filter base $\mathfrak{B}(\mathfrak{x})$ associated with $\mathfrak{x}$. Now it is not difficult to show directly that if x_0 is a cluster point of $\mathfrak{x}$, then some subnet of $\mathfrak{x}$ converges to x_0. The direct demonstration requires a construction very similar to (but somewhat more technical than) the proof of Proposition 2.5. However, from the view-point of filter theory, it is truly obvious that the collection $\mathfrak{D}$ of sets of the type $E\cap U$, where $E\in\mathfrak{B}(\mathfrak{x})$ and U is any neighborhood of x_0, forms a filter base which is a refinement of $\mathfrak{B}(\mathfrak{x})$ and which converges to x_0. Applying this comment and Proposition 2.5, we obtain immediately the existence of a subnet of $\mathfrak{x}$ which converges to x_0.

2.7 *Example.* We now indicate how nets arise quite naturally in certain indirect arguments. Let X and Y be topological spaces and $f:X\to Y$. Then the following statements are equivalent:

(A) f is continuous at x_0;
(B) if $\mathfrak{B}=\{E\}$ is a filter base which converges to x_0, then $\mathfrak{C}=\{f(E)\}$ converges to $f(x_0)$;
(C) if $\mathfrak{x}=\{x_\alpha\}$ is a net which converges to x_0, then $\mathfrak{y}=\{f(x_\alpha)\}$ converges to $f(x_0)$.

The proof is as follows: Let (A) be true, let V be a neighborhood of $f(x_0)$ and let U be a neighborhood of x_0 such that $f(U)\subseteq V$. Then since $\mathfrak{B}$ is ultimately in U, it follows that $\mathfrak{C}$ is ultimately in V, which proves (B). That (B) implies (C) follows immediately on passing to the filter base $\mathfrak{B}(\mathfrak{x})$ associated with $\mathfrak{x}$. Now let (C) hold. If (A) is false, there exists a neighborhood V of $f(x_0)$ such that every neighborhood U of x_0 contains at least one point x_U with $f(x_U)\notin V$. It is obvious that $\{x_U\}$ is a net which converges to x_0, so the contradiction is obtained, and

the equivalence established. (It should be mentioned that it is equally simple to show that (B) implies (A); the point was to illustrate that the negation of a statement concerning a filter base—here the filter base of neighborhoods of x_0—leads in a natural way to a net.)

3. Ultimate notions and compactness. We now come to two corresponding concepts that are of utmost utility in their respective convergence theories.

A net in a set X is said to be a *universal net*, or a *U-net*, if for any subset E of X, the net is either ultimately in E or its complement E'.

A filter base in a set X is said to be an *ultrafilter base*, or a *U-filter base*, if for any subset E of X, the filter base is either ultimately in E or in E'.

If $\mathfrak{B}$ is a U-filter base and if $\mathfrak{U}$ is the filter generated by $\mathfrak{B}$, then it is readily shown that $\mathfrak{U}$ has the following maximal property: if $\mathfrak{F}$ is any other filter in X such that $\mathfrak{F} \supseteq \mathfrak{U}$, then $\mathfrak{F} = \mathfrak{U}$. (This is a manifestation of the uniqueness properties possessed by filters, and not by nets.)

J. L. Kelley has proved the important fact that any net has a subnet which is a U-net. While his proof is short and not difficult, it is by no means obvious. On the other hand, we feel that the corresponding statement for filters is entirely elementary.

3.1. PROPOSITION. *If $\mathfrak{B}$ is a filter base in a set X, then there is a U-filter base $\mathfrak{U}$ which is a refinement of $\mathfrak{B}$.*

Proof. If $\mathfrak{B} = \{E_\alpha\}$ is not ultimately in a subset A of X, then by Proposition 1.1, either $\{A \cap E_\alpha\}$ is a filter base or $\{A' \cap E_\alpha\}$ is. By a standard transfinite argument we may continue this refinement process until we obtain a U-filter base.

We now examine the constructions in the preceding section in the light of these new concepts.

3.2. PROPOSITION. (a) *If $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$ is a U-net in a set X, then the associated filter base $\mathfrak{B}(\mathfrak{x})$ is a U-filter base.*

(b) *If $\mathfrak{B}$ is a U-filter base in X, then any associated net $\mathfrak{x}(\mathfrak{B})$ is a U-net.*

Proof. Let E be an arbitrary set in X; then either $x_\alpha \in E$ or $x_\alpha \in E'$ for $\alpha \geqq \alpha_0$. Consequently, the set $E(\alpha) = \{x_\lambda : \lambda \geqq \alpha\}$ is either in E or E' when $\alpha \geqq \alpha_0$, so (a) is proved. To prove (b), let $\mathfrak{x}(\mathfrak{B})$ be constructed as in 2.3. Since $\mathfrak{B}$ is a U-filter base it is ultimately in a given set E or in E' and the same follows for $\mathfrak{x}(\mathfrak{B})$.

As a consequence, we obtain Kelley's result:

3.3. PROPOSITION. *Every net has a universal subnet.*

Proof. If $\mathfrak{x}$ is a net in X, then by 3.1 the associated filter base $\mathfrak{B}(\mathfrak{x})$ has a refinement which is a U-filter base. The conclusion then follows from 2.5 and 3.2(b).

Although it is not appropriate to develop here the theory of U-nets and U-filter bases, a final application is in order. We recall that a topological space X

is *compact* (=bicompact) if every covering of X by open sets contains a finite subcovering.

3.4. PROPOSITION. *The following statements are equivalent*:

(a) *X is a compact topological space*;

(b) *if $\{F_\alpha\}$ is a class of closed subsets such that every finite collection has a non-void intersection, then the entire class has a non-void intersection*;

(c) *every filter base in X has a cluster point*;

(d) *every filter base in X has a refinement which converges to an element of X*;

(e) *every U-filter base in X converges to an element of X*;

(f) *every U-net in X converges to an element of X*;

(g) *every net in X has a subnet which converges to an element of X*;

(h) *every net in X has a cluster point.*

Proof. The fact that (a) and (b) are equivalent follows by a familiar application of deMorgan's laws and will be omitted. We now prove that each statement (b), $\cdots$, (g) implies the following one and that (h) implies (b). That (b) implies (c) follows immediately from the observation that if $\mathfrak{B}=\{E_\alpha\}$ is a filter base, then every point in $\cap_\alpha \overline{E}_\alpha$ is a cluster point. The implication (c)$\rightarrow$(d) was proved in Example 2.6. Proposition 3.1 trivially shows that (d) implies (e). Proposition 3.2(a) and Corollary 2.4 show that (e) implies (f). That (f)$\rightarrow$(g) follows directly from Proposition 3.3. If (g) holds, then (h) must, since a limit point of any subnet is a cluster point of the original net. Finally, let (h) be valid and $\mathfrak{B}=\{F_\alpha\}_{\alpha\in A}$ be a class of closed subsets every finite collection of which has a non-void intersection; then $\mathfrak{B}$ is a filter base. Let $\mathfrak{x}(\mathfrak{B})$ be any net associated with $\mathfrak{B}$. By (h) there is some point x_0 such that $x_0\in F_\alpha$, for every $\alpha\in A$. Thus (b) is true and the proof is complete.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the fifteenth William Lowell Putnam Mathematical Competition held on March 5, 1955, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Everett C. Dade, David Mumford and Kenneth Wilson; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Marcus T. Grisaru, Barry M. Mitchell and Charles B. H. Watson; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Yale University, New Haven, Connecticut. The members of the team were Morton E. Harris, John S. Lew and John G. Thompson; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Kenyon College, Gambier, Ohio. The members of the team were Trevor Barker, Thomas Jenkins and Robert Mosher; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Trevor Barker, Kenyon College; Everett C. Dade, Harvard University; David Mumford, Harvard University; Howard C. Rumsey, California Institute of Technology; and Jack Towber, Brooklyn College. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are John S. Lew, Yale University; R. J. Parikh, Harvard University; Hugh N. Pendleton 3d, Carnegie Institute of Technology; William G. Strang, Massachusetts Institute of Technology; and Kenneth Wilson, Harvard University.

The following teams, named in alphabetical order, won honorable mention: Brooklyn College, Brooklyn, New York, the members of the team being Toby Spiselman, James Thurber and Jack Towber; California Institute of Technology, Pasadena, California, the members of the team being Norman W. Albright, David G. Cantor and Howard C. Rumsey; Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being David C. Larson,

Hugh N. Pendleton 3d and George B. Rybicki; and New York University, New York, New York, the members of the team being Donald Fredkin, Charles Kahane and (Mrs.) Tilla Klotz.

Ten individuals were given honorable mention. The names are listed in alphabetical order: David G. Cantor, California Institute of Technology; William Hanf, University of California at Berkeley; Barry M. Mitchell, University of Toronto; Paul Monsky, Swarthmore College; Paul Payette, Université de Montreal; Donald L. Reinken, Princeton University; John R. Stallings, University of Arkansas; Donald Stevens, Cornell University; John G. Thompson, Yale University; and Charles B. H. Watson, University of Toronto.

A total of 361 individuals from 70 institutions entered the Competition this year. Of this number 105 individuals and seven institutions were unable to compete, due to various reasons. Therefore, a total of 256 undergraduates from 63 institutions took part in the Competition.

The following is a list of all colleges and universities which entered teams in the Competition. The list is arranged in alphabetical order: Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, Arizona State College, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Columbia University, Cornell University, Harvard University, Iowa State College, Kent State University, Kenyon College, Knox College, Knoxville College, Lebanon Valley College, Lewis and Clark College, Massachusetts Institute of Technology, New York University, Polytechnic Institute of Brooklyn, Princeton University, Queens College (Flushing, N. Y.), Queen's University (Kingston, Ont.), Reed College, Ricks College, Rose Polytechnic Institute, Rutgers University, Stanford University, Swarthmore College, Syracuse University, The Cardinal Stritch College, The College of the City of New York, The College of the Holy Cross, The Cooper Union, The Ohio State University, The United States Naval Academy, University of Arizona, University of Arkansas, University of British Columbia, University of California (at Los Angeles), University of Florida, University of Manitoba, Université de Montreal, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Pennsylvania, University of Rochester, University of Santa Clara, University of Toronto, University of Washington, Ursinus College, Virginia Union University, Washington University (St. Louis, Mo.), Wayne University, and Yale University.

The following colleges and universities entered individual contestants only: Barry College, Bethany College, Blackburn College, Boston University, Haverford College, Sacramento State College, Saint Olaf College, State University of New York, College of Forestry (Syracuse, N. Y.), University of California (at Berkeley), University of Nebraska, University of Omaha, and University of Texas.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative

rankings of the first five will not be divulged) by writing to the Director of the Competition, Room 301 Merrill Hall, Kent State University, Kent, Ohio. These rankings may now be given to the individual contestants by their own departments of mathematics. Any other departments of mathematics may obtain the individual rankings of contestants for the purpose of selecting graduate students.

Those participating in the Competition were given the following lists of problems:

MORNING SESSION:

Omit one question.

1. Prove that there is no set of integers m, n, p except 0, 0, 0 for which $m+n\sqrt{2}+p\sqrt{3}=0$.
2. $A_1A_2 \cdots A_n$ is a regular polygon inscribed in a circle of radius r and center O. P is a point on line OA_1 extended beyond A_1. Show that

$$\prod_{i=1}^{n} \overline{PA_i} = \overline{OP}^n - r^n.$$

3. Suppose that $\sum_{i=1}^{\infty} x_i$ is a convergent series of positive terms which monotonically decrease (that is, $x_1 \geqq x_2 \geqq x_3 \geqq \cdots$). Let P denote the set of all numbers which are sums of some (finite or infinite) subseries of $\sum_{i=1}^{\infty} x_i$. Show that P is an interval if and only if

$$x_n = \sum_{i=n+1}^{\infty} x_i \qquad \text{for every integer } n.$$

4. On a circle n points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent (except at the endpoints), how many points of intersection are there?
5. If a parabola is given in the plane, find a geometric construction (ruler and compass) for the focus.
6. Find a necessary and sufficient condition on the positive integer n that the equation

$$x^n + (2+x)^n + (2-x)^n = 0$$

have a rational root.

7. Consider the function f defined by the differential equation

$$f''(x) = (x^3 + ax)f(x)$$

and the initial conditions $f(0)=1$, $f'(0)=0$. Prove that the roots of f are bounded above but unbounded below.

AFTERNOON SESSION:

Omit one question.

1. A sphere rolls along two intersecting straight lines. Find the locus of its center.
2. Suppose that f is a function with two continuous derivatives and $f(0)=0$. Prove that the function g, defined by $g(0)=f'(0)$, $g(x)=f(x)/x$ for $x\neq 0$, has a continuous derivative.
3. Prove that there exists no distance-preserving map of a spherical cap into the plane. (Distances on the sphere are to be measured along great circles on the surface.)
4. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?
5. Given an infinite sequence of 0's and 1's and a fixed integer k. Suppose that there are no more than k distinct blocks of k consecutive terms. Show that the sequence is eventually periodic. (For example, the sequence 1101101-0101 followed by alternating 0's and 1's indefinitely, which is periodic beginning with the fifth term.)
6. Prove: If $f(x)>0$ for all x and $f(x)\to 0$ as $x\to\infty$, then there exists at most a finite number of solutions of

$$f(m)+f(n)+f(p)=1$$

in positive integers m, n, and p.

7. Four forces acting on a body are in equilibrium. Prove that, if their lines of action are mutually skew, they are rulings of an hyperboloid.

Solutions of the Problems*

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The presentation here is intended as a brief sketch of the method of proof rather than as a model of a detailed proof such as is expected from the contestants.

PART I

1. By transposing and squaring $3p^2-m^2-2n^2=2mn\sqrt{2}$ and since $\sqrt{2}$ is irrational, $mn=0$. If $n=0$ then $-m=p\sqrt{3}$, and since $\sqrt{3}$ is irrational, $p=m=0$. If $m=0$ then $n\sqrt{2}+p\sqrt{3}=0$ and $2n=-p\sqrt{6}$, and since $\sqrt{6}$ is irrational, $p=n=0$.

2. Consider the plane of the polygon as the complex plane with O as zero and A_1 as r. Using the same notation for the complex coordinate of a point as for the point, we have

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

$$\prod_{i=1}^{n} \overline{AP}_i = \prod_{i=1}^{n} \left| P - r \times \exp(2\pi(i-1)/n) \right| = \left| \prod_{i=1}^{n} (P - r \times \exp(2\pi(i-1)/n)) \right|$$
$$= \left| P^n - r^n \right| = P^n - r^n.$$

3. If $x^k = 2p + \sum_{i=k+1}^{\infty} x_i$ with $p > 0$ and if $x_k - p = p + \sum_{i=k+1}^{\infty} x_i = \sum_{i=1}^{\infty} x_i\theta_i$ with each θ_i either zero or one, then $\theta_i = 0$, $i = 1, 2, \cdots, k$, by the monotoneity of the terms, but then $\sum_{i=1}^{\infty} x_i\theta_i \leqq \sum_{i=k+1}^{\infty} x_i = x_k - 2p$ gives a contradiction. Thus the condition is necessary.

If $x_n \leqq \sum_{i=n+1}^{\infty} x_i$, $n = 1, 2, \cdots$, and if $0 < p < \sum_{i=1}^{\infty} x_i$, take i_1 as the least positive integer n for which $x_n \leqq p$. Assuming inequality occurs, take j_1 as the largest positive integer n such that $S(i_1, j_1) \leqq p < S(i_1, j_1+1)$, where by $S(m, n)$ we mean the sum $\sum_{i=m}^{n} x_i$. Such a choice of j_1 is possible since $S(i_1, \infty) \geqq x_{i_1-1} > p$ if $i_1 > 1$ and $S(1, \infty) > p$ if $i_1 = 1$. If k blocks of consecutive terms have been chosen such that $i_h < j_h < i_{h+1} < j_{h+1}$, $h = 1, 2, \cdots, k-1$, and $0 \leqq p - \sum_{r=1}^{k} S(i_r, j_r) < x_{i_k+1}$ and if equality does not obtain in the last formula then take i_{k+1} as the smallest positive integer n such that $x_n \leqq p - \sum_{k=1}^{k} S(i_r, j_r)$ and j_{k+1} as the largest positive integer n such that $0 \leqq p - \sum_{r=1}^{k+1} S(i_r, j_r) < x_{j_k+1+1}$. The existence of i_{k+1} and j_{k+1} with the required properties follows as before. The process either terminates if an equality occurs or produces an infinite subseries each partial sum of which is less than p. Since any partial sum which terminates in the k-th group of terms exceeds $\sum_{r=1}^{k-1} S(i_r, j_r) > p - x_{j_{k-1}+1}$, the series converges to p.

4. Every subset of four distinct points of the given points determines an intersection point within the circle and conversely. Thus the number of intersections within the circle is ${}_nC_4$.

5. The midpoints of any two parallel chords determine a line L parallel to the axis. The axis is then constructed as the perpendicular bisector of a chord which is perpendicular to L. This determines the vertex V. A line through V making a 45° angle with the axis intersects the parabola in a point P. If Q is the orthogonal projection of P on the axis then the focus F is determined by $\overline{VQ} = 4\overline{VF}$.

6. Clearly n must be odd. If expanded, the leading term of the polynomial is x^n and the constant term is 2^{n+1}. Thus every rational root is in the set $\pm 2^k$, $k = 0, 1, \cdots, n+1$. The positive values are excluded since if x is positive $(x+2)^n > (x-2)^n$. Since $k = 0$ makes all three terms odd, this case is also excluded. Substituting $x = -2^k$, $1 \leqq k \leqq n+1$, and dividing by 2^n we have $2^{n(k-1)} = (2^{k-1}+1)^n - (2^{k-1}-1)^n = 2\sum_{j=0}^{n-1} (2^{k-1}+1)^j (2^{k-1}-1)^{n-1-j}$. The sum on the right is clearly odd and thus $n(k-1) = 1$, and $n = 1$ and $k = 2$. Substitution verifies the sufficiency.

7. Let $f(x_0) = f'(x_0) = 0$. Then if $|x^3 + ax| \leqq M$ for $|x| = 1 + |x_0|$ and if $K = \max |f(x)|$ for $|x - x_0| \leqq 1/(1+\sqrt{M})$, then in this last interval $|f(x)|$

$=|f''(\theta x)|\,|x-x_0|^2/2$, with $0<\theta<1$. Taking x to maximize the left member in $|x-x_0| \leqq 1/(1+\sqrt{M})$ we have $K \leqq MK/2(1+\sqrt{M})^2 < K/2$ and thus $K=0$, and f vanishes identically in an interval about x_0 of half-length $1/(1+\sqrt{M})$. The same argument may then be applied to the end point of this interval which is nearest the origin to extend the interval by the same amount. By finite induction, since the length of the interval does not decrease, $f(0)=0$, which is absurd. Thus f and f' do not vanish together and the zeroes of f are isolated. This results also from standard existence and uniqueness theorems of the theory of differential equations.

If consecutive zeroes of f occurred in a region where x^3+ax was positive, then f positive (negative) in this interval implies the graph of f is concave upward (downward) in the interval, and this is absurd. Thus the zeroes are bounded above.

Now suppose $x^3+ax<-1$ for $x \leqq x_0$ and suppose $f(x_0)=k>0$. Then f' must become positive somewhere to the left of x_0, for to deny this would require $f(x) \geqq k$ for $x<x_0$, and then $f'(x)>f'(x_0)+k(x_0-x)$ which contradicts the denial. But if $f'(x_1)>0$, $x_1<x_0$, then $f'(x) \geqq f'(x_1)$ for all $x<x_1$ until a zero of f is reached. Thus a zero of f exists to the left of x_0. A similar argument applies if $f(x_0)=k<0$, and thus f has an infinite number of negative zeroes.

PART II

1. Take the xy-plane as the plane of the intersecting lines and the positive x-axis as bisector of the angle formed by the rays in contact with the sphere. Let P: $(x, 0, z)$ be the center of the sphere and then Q: $(x, 0, 0)$ is the center of the circle in which the xy-plane cuts the sphere. The points of contact of the rays, T_1 and T_2, are distant from Q by $x \sin\theta$ where 2θ is the angle between the rays. From the right triangle PQT_1 we have $x^2\sin^2\theta+z^2=R^2$ with R the radius of the sphere. The complete locus is thus a pair of ellipses.

2. For $x \neq 0$, $g(x)=f(x)/x=f'(\theta x)$ with $0<\theta<1$, and since f' is continuous $\lim_{x\to 0} g(x)=f'(0)=g(0)$. Thus g is continuous at the origin and so at all points. $\lim_{x\to 0}\ [g(x)-g(0)]/x=\lim_{x\to 0}\ [f(x)-xf'(0)]/x^2=\lim_{x\to 0}\ [f'(x)-f'(0)]/2x$ $=f''(0)/2$ by L'Hospital's rule. For $x=0$, $g'(x)=[xf'(x)-f(x)]/x^2$ and by L'Hospital's rule the limit as $x\to 0$ is also found to be $f''(0)/2$. Thus g' is continuous at $x=0$ and at other points by standard theorems of calculus.

3. Under a distance preserving map the length of curves is unaltered. A small circle in the spherical cap with spherical distance $R\theta$ from the circumference to the center must transform into a circle in the plane of radius $R\theta$. The perimeter of the circle on the sphere is then $2\pi R \sin\theta$ and for the image circle is $2\pi R\theta$. Since $\sin\theta<\theta$ for $\theta>0$, the contradiction follows.

4. If $j+i$ is divisible by p_i^2, $i=1, 2, \cdots, k$, then so is $j+i+rP_k$ for any integer r and with $P_k=\prod_{i=1}^{k} p_i^2$. If p_{k+1} is relatively prime to P_k, then there are

integers s and t such that $tp_{k+1}^2+sP_k=1$. By squaring if necessary, we may assume $s>0$. Take a positive integer u such that $v=up_{k+1}^2-j-k-1>0$ and then $vsP_k+j+k+1=(u-vt)p_{k+1}^2$. Thus we see $j+i+vsP_k$ is divisible by p_i^2, $i=1, 2, \cdots, k+1$.

5. Let S_j denote the finite sequence $a_{j+1}, a_{j+2}, \cdots, a_{j+k}$ with a_n the n-th term of the given sequence. Let r be the least positive integer such that for some j, S_j and S_{j+r} are identical. Then the S_i are distinct for $i=j, j+1, \cdots, j+r-1$ and so $0<r\leqq k$. Assume $r<k$. The sequence of r digits $a_{j+1}, a_{j+2}, \cdots, a_{j+r}$, which we denote by R_j, is then identical with the sequence R_{j+r}. By finite induction it follows readily that $a_{j+h}=a_{j+h+r}$, $h=1, 2, \cdots, k$. The minimal property of r requires that the R_i be distinct for $i=j, j+1, \cdots, j+r-1$. Assume now that $a_{j+k+r+1}\neq a_{j+k+1}$. Some two of $S_{j+1}, S_{j+2}, \cdots, S_{j+k+r}$ are identical. But if S_{j+h} and S_{j+k+s} are in this set and identical, then s is a multiple of r, for otherwise the first r digits differ. But if $s=mr$ then $a_{j+k+r+1}$ is the $[k-(m-1)r-h+1]$-st digit of S_{j+h+mr} and the corresponding digit of S_{j+h} is $a_{j+k-(m-1)r+1}$ which is different. But this is a contradiction and thus $a_{j+h}=a_{j+h+r}$, $h=1, 2, \cdots, k+1$, so that the pattern persists. Since the same hypotheses now apply to S_{j+1} and S_{j+r+1}, the pattern must persist another step, and by induction $a_{j+h}=a_{j+h+r}$ for all positive integers h. If $r=k$ then $S_{j+1}, S_{j+2}, \cdots, S_{j+k}$ are distinct by the minimal property of $r=k$. S_{j+k+1} differs from S_{j+1} in the last place if $a_{j+2k+1}\neq a_{j+k+1}$, and differs from the others because of the minimal property of $r=k$. Thus the S_i, $i=j+1, \cdots, j+k+1$, are distinct, and the contradiction proves $a_{j+2k+1}=a_{j+k+1}$ and the proof follows as before.

6. Take i_1 such that $i>i_1$ implies $f(i)<1/3$. Let $k=\text{Max } f(i)$ for $i\leqq i_1$ and $f(i)<1$. If the set of such i is empty no solution exists. Otherwise $k<1$ and we take $i_2>i_1$ so that $i>i_2$ implies $f(i)<(1-k)/2$. Let $h=\max\left[f(i)+f(j)\right]$ $i\leqq i_2$, $j\leqq i_2$, and $f(i)+f(j)<1$. If no such pairs i and j exist then no solution exists. Otherwise $h<1$ and we take $i_3>i_2$ so that $i>i_3$ implies $f(i)<1-h$. Let m, n, p be any solution and we suppose $f(m)\geqq f(n)\geqq f(p)$. Then $f(m)\geqq 1/3$ and thus $m\leqq i_1$. Hence $f(n)+f(p)=1-f(m)\geqq 1-k$ and thus $f(n)\geqq(1-k)/2$ and $n\leqq i_2$. Finally $f(p)=1-f(m)-f(n)\geqq 1-h$ and so $p\leqq i_3$.

7. Let the lines of action of the forces be L_i, $i=1, 2, 3, 4$. Any line L which intersects three of these must also intersect the fourth since the moments about L must vanish. Let L_5, L_6, L_7 be distinct lines each of which intersect all four of the lines of action of the forces. Moreover L_5, L_6, and L_7 must be mutually skew and thus determine a regulus of lines, *i.e.*, all lines which intersect all three, and the lines of this regulus are rulings of an hyperboloid. For the construction of the equation of the hyperboloid and further discussion see Bell, Robert, *Coordinate Geometry of Three Dimensions*, Macmillan, 1954, pp. 163–165.

MATHEMATICAL CONSULTANTS, COMPUTATIONAL MATHEMATICS AND MATHEMATICAL ENGINEERING

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1. Introduction. This note, and a companion (which may eventually appear on *The teaching of concrete mathematics*) are after-effects of the Conference on Training in Applied Mathematics held in New York on October 22–24, 1953.* They represent an attempt to combine, in unknown proportions, some implicit conclusions of the conference, as I sensed them, and some very personal feelings which have developed slowly over a longer period. The present note discusses attitudes which mathematicians and mathematics departments might take toward (1) the training of mathematical consultants, (2) the place of computational mathematics, (3) the place and role of groups working in theoretical mechanics, and (4) the emergence of Departments of Mathematical Engineering and of training in computation engineering.

Before going on, a word about the word "engineering." Some think it is a bad word—they like to be pure. Some think it is a good word—they like to be engineers. The writer associates with both pure mathematicians and engineers regularly and by choice—he thinks of it as a neutral word, and he hopes that his readers will do the same, at least while reading through and mulling over this discussion.

2. Mathematical consultants. Probably the most important single mathematical function in industry and government is that of the mathematical consultant. Today there are few mathematical consultants, tomorrow there will not be very many, but their importance will far outweigh their numbers.

On their training the writer has strong views, and, he believes, strong support for them. First, and foremost, it should be a research training, most commonly in pure mathematics, but not infrequently in physics, a mathematized branch of engineering, or even in applied mathematics. Secondly, it should involve as wide a background as possible.

The basic requirements for a mathematical consultant are threefold. In addition to the research training and the broad mathematical background, the mathematical consultant needs, as do consultants based on any scientific field, *above all else* an interest in the other man's problems—in these problems as wholes, not just in their mathematical aspects.

The future consultant will be trained along the same lines as the future research mathematician, and will be distinguished from him, not by a lack of research ability (for he needs as much as they do), but rather by his broader interests. His training asks only of Departments of Mathematics that:

(1) they encourage their students to browse around, and

* Bull. A.M.S., vol. 60, 1954, pp. 38–44.

(2) they consider it reasonable that some of their best students become mathematical consultants.

For mathematical *consultants* we don't want mathematical engineers, we *do want mathematicians* (or theoretical physicists turned mathematicians, or electrical engineers turned mathematicians, *etc.*)!

3. The place of computational mathematics. The writer makes a sharp and important distinction between the mathematics and the engineering of computation. The mathematics of computation is concerned with what happens as a result of various computational steps or procedures. The engineering of computation is concerned with how best to choose steps or procedure to obtain the desired results with relatively small expenditures of effort or money.

Thus, for example, computational mathematics recognizes *one* interpolating polynomial through a given set of points, and notes in passing that values of this polynomial can be found by Aitken's method, by forward or backward Newton's formula, by Everett's formula, and by many other specific expressions. Computation engineering is concerned with the advantages and disadvantages of the various methods, and with the choice between them in particular situations.

Computational mathematics is a branch of mathematics like any other, and is logically taught by mathematicians. Today many places teach a little at the undergraduate level, mostly mixed with considerable computation engineering. Tomorrow, a few institutions will have graduate programs preparing a few Ph.D.'s in computational mathematics. These Ph.D.'s will need to meet the requirements of mathematical ability and background currently required of Ph.D.'s in pure mathematics, and many of them will go on to do research in computational mathematics as teachers in colleges and universities.

The responsibilities of mathematics departments toward computational mathematics are, then, to be met by:

(1) readiness, nay even eagerness, to teach computational mathematics at the undergraduate level to all who wish to learn (in all departments), and

(2) provision of needed research support, which is needed by both computation engineering and the further development of computational mathematics, through the training of a few good Ph.D.'s in computational mathematics. (The need exists now, and will grow rapidly. The Ph.D.'s should fully meet the standards of pure mathematics. A few institutions could meet the present needs.)

4. Theoretical mechanics. To allow for the mental habits of many readers, we shall use the term "theoretical mechanics" for what would be better called, in the writer's opinion. "(classical) physical mathematics," where the order of the words (just as in "physical chemistry" or in "mathematical physics") implies that the subject is *mathematics* which is oriented toward physics. Besides

all those things which are truly mechanics we include here electromagnetic theory and boundary value problems.

In view of the prevalence of the relation

applied mathematics≡theoretical mechanics (modulo unimportant details)

in many minds, no discussion bearing on the applications of mathematics can avoid a discussion of theoretical mechanics sufficient for orientation. (In England the congruence of applied mathematics with theoretical mechanics seems to have been strengthened to an equality.) According to the recent NRC-AMS conference on applied mathematics in New York City, the recipe for theoretical mechanics in the United States seems to be:

2 parts mathematics, 1 part physics, 2 parts engineering, mix well and serve labelled "Applied Mathematics."

Why should this recipe be preferred?

The writer's estimate runs about as follows: At the present time, theoretical mechanics tends to be too routine for the science departments (physics and mathematics) and is almost surely too fundamental for the engineering departments (aeronautical, electrical, hydraulic, mechanical). Thus it has to be operated on an intermediate "applied science" level. In most institutions this is most easily done, if it is done at all, through a cooperative arrangement between science and engineering. This is quite likely to be a transient situation. If the engineering departments strengthen their fundamental side, as many of them indicate desires and plans to do, then theoretical mechanics may become a sophisticated branch of engineering. (After all, some engineers assert that, now the physicists have dived into the nucleus, they will have to take over not only theoretical mechanics, but experimental mechanics, optics, acoustics and the rest of classical physics. This seems unlikely in the long run.) Whether it will then be a branch of *mathematical* engineering or not is another matter, but if the ratio of mathematicians to physicists interested in theoretical mechanics continues to have its present high value, this will probably happen.

What, then, is the appropriate attitude of the mathematics department toward theoretical mechanics? As far as an undergraduate program is concerned, the answer is clear. It is best for the mathematics department to teach mathematics, for the physics department to teach physics and for the engineering departments to teach engineering. Unless there is an unusually strong base in the engineering department, a base so strong as to be extremely rare today, such a program is probably best guided by a joint committee. And the recipe: 2 parts mathematics, 1 part physics, 2–3 parts engineering, seems entirely natural for such a committee. This can be considered, without loss of generality for *internal* discussion among mathematicians, a program in mathematical engineering. (For discussion with other departments it will undoubtedly be necessary to emphasize the joint aspects.) As such it poses no new problems beyond those discussed in the next section.

At the graduate and postgraduate level, the situation is different. Here there

must be strong research interests in theoretical mechanics. These research interests are not (with rare exceptions) housed in engineering, in physics or in mathematics. There must be a house for them somewhere. Hence the development of graduate institutes in the area (*e.g.*, Brown, Indiana, Maryland). Toward such a group what should be the attitude of the mathematics department? Clearly those groups function, at those instants when they function mathematically, at a more fundamental level than, at present at least, any of the conventional engineering departments. Thus they are closer relatives. On the other hand, they are not in the immediate family. The appropriate relation would seem to be that of double first cousins—very close, but not quite inside the family. (Hence one must be doubly careful of the Persian proverb "To hate like cousins!")

The most obvious outward implications of such an attitude would seem to be these:

(1) encouragement of joint appointments whenever the appointee would fit into the mathematics department;

(2) acquiescence, for the present, in a title for such a group involving the words "applied mathematics" if that is what the group wishes,

(3) continuous, but gentle, education about the advantages to both parties of a title which expresses the character of the group more clearly and accurately.

In brief: support and cooperation but avoidance of blending.

5. Science, mathematics and engineering. Every mathematician acquainted with the history of United States colleges and universities which have had two mathematics departments, whether the second be called a department of applied mathematics or a department of engineering mathematics, fears such arrangements. For in almost every case, they have led to watering down the quality of mathematics and to ill feeling. There is a natural tendency to carry over this feeling to any administrative arrangement where all the "mathematics" is not in one mathematics department. This natural tendency may not always be wise, as we shall try to show.

The writer believes, indeed, that chemistry departments have gained rather than lost from the presence of chemical engineering departments and that physics departments have gained rather than lost from the existence of mechanical, electrical, and aeronautical engineering departments. He believes, furthermore, that most members of chemistry and physics departments would agree to this. Let us try to inquire into why this is so.

There was then, and surely continues today, a serious need for chemical, mechanical, electrical and aeronautical engineers. Three ways of meeting this need are conceivable (we shall state them in terms of physics alone for convenience):

(1) The physics department could have taught both physics and engineering.

(2) The physics department could have taught physics and the engineers could have taught engineering, or

(3) The engineers could have taught both physics and engineering.

Of these, (2) was generally selected. The writer submits that this was the best choice for the physics department. Why?

If (1) had been selected, what would have happened to physics departments? They would have faced the teaching of courses in alternating current machinery, bridge design and propeller theory, to name some examples. Such courses would have been boring to the permanent staff, but would have required too much background to be safely turned over to fresh Ph.D.'s. Either the morale and research potential of the permanent staff or the reputation of the training would have gone down, down, down.

If (3) had seen selected, what would have happened to physics departments? At first glance, the main penalty would have been a size and support penalty due to a loss of students. But actually the situation would have been far worse in the long run. The wholly engineer-taught engineer would in the long run have lost out on fundamental physics. And in the longer run, this would have been evident in comparison with the increasing demands placed on the engineer. As a consequence, the physics departments would have to shoulder the responsibility of providing more fundamentally trained engineers and the difficulties associated with (1) would not have been avoided.

Either (1) or (3) would have a fate worse than (2) *for physics departments*—and, incidentally, either would, of course, have provided engineers with poorer training. Perhaps this example has a moral for mathematics departments!

This moral, if it exists, must be this: "When the demand for mathematical engineers begins, plan to meet it cooperatively—plan to teach the relevant mathematics in the mathematics department—plan to teach the mathematical engineering elsewhere." The writer believes that this moral exists and is important. He hopes that mathematics departments will preserve both their research potential and their student-semesters of classes by adopting this policy when it becomes timely.

6. What is mathematical engineering? Many readers may by now be muttering under their breaths, "But what *is* mathematical engineering?". To this question we have no complete answer, any more than Faraday could explain the uses of newly discovered electromagnetism. But we can and will give partial answers.

Mathematical engineering consists of those branches of engineering where the single most important tool is mathematics. This is the natural definition. Does it help us? Certainly it helps us a little, for it makes it clear that the engineering of computation is certainly a branch of mathematical engineering, for surely in planning computation—in planning mathematics—mathematics is the one most essential tool. In those parts of theoretical mechanics too, where the physics and the differential equations are long known and firmly settled, the most important single tool may well be mathematics, and, where it is, these parts of theoretical mechanics may prove to be mathematical engineering. (In the area now labelled industrial engineering, some further branches are probably

being shaped, but it would be premature to try to identify them here.)

First to be of importance is undoubtedly that which is first above—computation engineering. It is here that the first great needs will come.

7. Computation engineering. Today the need for mathematical engineers has begun to appear. The first need is for computation engineers! The arrival of the first batch of IBM type 701 high-speed calculators (formerly known as Defense Calculators, latterly as Electronic Data Processing Machines) is a substantial indication. The estimate of a staff of 30 persons per machine for economic balance is almost certainly conservative. But this means that over 500 trained persons are required for these machines alone.* What kind of persons?

Today the man with a 701 seeks out A.B.'s or M.A.'s in mathematics for his coders and lower level problem analysts, who will form 90% or more of his staff. The work he has for them is high-level routine and requires a solid grasp of a substantial amount of mathematics. Qualitatively, it seems to the writer entirely comparable to the work of an average mechanical or electrical engineer. As time goes on, are not the increasing numbers of such jobs going to be filled by mathematical engineers, whatever they may be called? How can it be otherwise?

Today the demand for computation engineers is definitely here. Tomorrow it will be larger. Today a few institutions will set up sources of supply. (How many? This is not clear.) Tomorrow, more and more will join them. (How fast? This is far from clear.) What should mathematics departments do, or be prepared to do?

If the moral drawn above is correct, then mathematics departments should, in the writer's judgment, act as follows:

(1) They should prepare to cooperate in the setting up of training for computation engineers under engineering auspices, and

(2) They should prepare to teach the necessary mathematics, *including the mathematics of computation*, in the mathematics department.

8. Summary. In the writer's opinion, then, Departments of Mathematics should:

(1) encourage at least some of their students to "graze around"

(2) look forward to a very few of their best students becoming mathematical consultants in industry or government

(3) stand ready to teach the mathematics of computation to all who wish to learn

(4) in a few cases, start training high-caliber Ph.D.'s in computational mathematics

(5) encourage joint appointments with theoretical mechanics groups whenever this is reasonable

(6) acquiesce, for the present, in a title of "applied mathematics" for such a group

* *Note added in proof.* In the year or year and a half since these words were first written, the predictable demand has increased by a *factor* of at least five, or more nearly ten!

(7) point out the advantages to both parties, gently but steadily, of a more appropriate title
(8) prepare to cooperate in the setting up of training in computation engineering under engineering auspices, and
(9) prepare to teach the necessary mathematics in the mathematics department.

Through such policies can Departments of Mathematics maintain and increase their strength and quality.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—VI

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1. Introduction. In two recent notes, [1], [2], we have shown how various results relating to the determinant of A, a positive definitive matrix, could be derived from the well-known identity

$$\frac{c_n}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dV_n \tag{1}$$

where the integration is over all x, and $c_n = \sqrt{\pi}^n$, a constant depending only upon the dimension n.

If we define, for $k = 1, 2, \cdots, n$,

$$|A|_k = \prod_{i=1}^{k} \lambda_i \tag{2}$$

where λ_i, $i = 1, 2, \cdots, n$, are the characteristic roots of A, a positive definite matrix, arranged in increasing order of magnitude, it was shown by Ky Fan, [3], [4], that

$$|A\lambda + B\mu|_k \geqq |A|_k^{\lambda} |B|_k^{\mu}, \qquad \lambda, \mu \geqq 0, \lambda + \mu = 1. \tag{3}$$

This result, together with some additional results, was recently obtained in a different fashion by Oppenheim, [5].

The purpose of the present note is to establish an identity for $|A|_k$ similar to (1). This identity may then be utilized to derive (3) in the same manner that

(1) was used to obtain the particular case $k=n$ of (3) (*cf.* [1]), namely by a direct application of Hölder's inequality.

The identity is:

THEOREM.

$$\frac{\sqrt{\pi}^k}{|A|^{\frac{1}{k}}} = \operatorname*{Max}_L \int_L e^{-(x,Ax)} dV_k \tag{4}$$

where L is a k-dimensional subspace of the n-dimensional space.

Proof. It is easy to see that we may begin with the case where A is already in diagonal form, $(x, Ax)=\lambda_1 x_1^2+\lambda_2 x_2^2+\cdots+\lambda_n x_n^2$, since an orthogonal transformation leaves the formulation of the problem invariant. Hence we must show that

$$\frac{\sqrt{\pi}^k}{(\lambda_1\lambda_2\cdots\lambda_k)^{1/2}} = \operatorname*{Max}_L \int_L e^{-\lambda_1 x_1^2-\lambda_2 x_2^2-\cdots-\lambda_n x_n^2} dV_k. \tag{5}$$

Define $V_a(\rho)$ to be the volume contained in the k-dimensional region

$$\begin{aligned} \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 &\leqq \rho, \\ (x, \alpha_i) &= 0, \qquad i = 1, 2, \cdots, n-k. \end{aligned} \tag{6}$$

Then

$$V_a(\rho) = \rho^{k/2} V_a(1) \tag{7}$$

and we have

$$\int_L e^{-\lambda_1 x_1^2-\lambda_2 x_2-\cdots-\lambda_n x_n^2} dV_k = \int_0^\infty e^{-\rho} dV_a(\rho) = \left[\frac{k}{2}\int_0^\infty e^{-\rho}\rho^{(1/2)k-1} d\rho\right] V_a(1). \tag{8}$$

To complete the proof, we must show that the maximum of $V_a(1)$ is attained when the relations $(x, \alpha_i)=0$ are $x_{k+1}=x_{k+2}=\cdots=x_n=0$. This, however, is an immediate consequence of the formula for the volume of an ellipsoid in terms of the characteristic roots of the associated symmetric matrix and the separation theorem for characteristic roots furnished by the min-max characterization of the characteristic roots.

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APPROXIMATION BY ENUMERABLE SETS

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Let $\{r_n\}$ be an enumerable set of real numbers and let $\{d_n\}$ be a sequence of positive real numbers. We say that a set E of real numbers is *approximated by* $\{r_n\}$ *within* $\{d_n\}$ if, whenever ξ is in E, the inequality $|\xi - r_n| < d_n$ holds for infinitely many n.

For example, let the rationals be enumerated in the sequence

$$0,\ 1,\ 1/2,\ 1/3,\ 2/3,\ 1/4,\ 2/4,\ 3/4,\ 1/5,\ 2/5,\ \cdots, \tag{1}$$

so that $p/q = r_n$ with

$$n = p + 2 + (q-1)(q-2)/2.$$

The theorem of Dirichlet, that $|\xi - p/q| < 1/q^2$ for infinitely many q, gives

$$|\xi - r_n| < 1/(2n) \tag{2}$$

for infinitely many n. If duplicates are omitted in (1) then we have, asymptotically,

$$|\xi - r_n| < 3/(\pi^2 n)$$

upon using the fact that the Euler ϕ function satisfies

$$\phi(1) + \cdots + \phi(n) \sim 3(n/\pi)^2.$$

Suppose one replaces the sequence of rationals $\{r_n\}$ by another enumerable sequence constructed expressly for the purpose of getting a good approximation. How good an approximation can be obtained? It is known that (2) is nearly optimum when $\{r_n\}$ is the rational sequence. How nearly is it optimum in this broader sense? Such are the questions with which the present discussion is concerned. It will be seen that a rather complete answer can be given by very elementary methods.

Direct approximation. With regard to approximation of the type just considered, we have the following:

Remark 1. If $\sum d_n = \infty$, then there is an enumerable set $\{r_n\}$ (depending on $\{d_n\}$ but not on E) such that every set E is approximated by $\{r_n\}$ within $\{d_n\}$.

For proof, it suffices to lay off adjacent intervals of lengths $d_0, d_1, d_2, \cdots$ on the real axis. We start at the origin, proceed to the right until the point $+1$ is passed, then turn to the left until -1 is passed, then to the right until $+2$ is passed, then -2, $+3$, -3, and so on. Since $\sum d_n = \infty$, the process is possible; and the end points of the intervals yield the sequence $\{r_n\}$.

That the condition $\sum d_n = \infty$ is necessary is shown by the following:

Remark 2. Suppose $\sum d_n < \infty$. If there is an enumerable set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$, then the measure of E is zero.

For proof, construct an interval of length $2d_n$ centered at r_n. If N is fixed,

each point of E is interior to one of these intervals, with $n>N$. Hence E is covered by a set of intervals whose measure does not exceed

$$2d_N + 2d_{N+1} + 2d_{N+2} + \cdots;$$

and this is arbitrarily small.

This result shows that (2) is almost optimum; the set for which infinitely often

$$|\xi - r_n| < 1/n^{1+\epsilon}$$

with $\epsilon>0$ has measure zero. Alternatively, the set for which infinitely often

$$|\xi - p/q| < 1/q^{2+\epsilon}$$

has measure zero. Instead of $1/q^{2+\epsilon}$ one can use $f(q)/q$ where $\sum f(q)<\infty$. (Of course these well-known results can be obtained without use of the general theory.)

The following shows that one can deduce no more than $m(E)=0$, from the mere convergence of $\sum d_n$:

Remark 3. If E has measure zero, then there is an enumerable set $\{r_n\}$ and a sequence $\{d_n\}$ with $\sum d_n<\infty$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$.

For proof, cover E by a countable set of intervals $\{I_{n,1}\}$, then by a new set $\{I_{n,2}\}$, and so on. With $d_{n,k}=|I_{n,k}|$ we can carry out the construction in such a way that

$$\sum d_{n,k} < 1/2^k, \qquad k = 1, 2, \cdots.$$

The midpoints of these intervals yield an enumerable set, $\{r_{n,k}\}$; the associated intervals yield a sequence of lengths $\{d_{n,k}\}$. Since

$$\sum\sum d_{n,k} < 1,$$

the construction satisfies the required conditions. By enumerating $\{r_{n,k}\}$ suitably, we can make the corresponding sequence $\{d_{n,k}\}$ monotonic.

The preceding remark suggests the following:

Query 1. If $\{d_n\}$ is given with $\sum d_n<\infty$, is there a set E of measure zero which is not approximated within $\{d_n\}$ by any $\{r_n\}$?

Considering the Liouville numbers shows that the approximation can be arbitrarily good without our having E enumerable:

Remark 4. Given a sequence $\{d_n\}$ with $d_n>0$, there is a nonenumerable set E and an enumerable set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$.

This result suggests a question related to query 1:

Query 2. For each sequence $\{d_n\}$, $d_n>0$, suppose there is a corresponding set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$. Does it follow that E is enumerable?

Approximation in mean. Let $\delta(\xi)$ denote the distance from ξ to the nearest

point of a finite set $r_1, r_2, \cdots, r_n$. We write

$$D(E, r_i, n) = \int_E \delta(\xi) d\xi$$

as a measure of the accuracy with which E is approximated by the finite set $\{r_i\}$. If $\{r_i\}$ is infinite, we consider the behavior as $n \to \infty$. For simplicity, E is taken as the interval $[0, 1]$ henceforward.

Remark 5. We have $D(E, r_i, n) \geqq 1/(4n)$, and there is an enumerable set $\{r_n\}$ for which this bound is attained infinitely often as $n \to \infty$.

For proof, we observe that D is a continuous function of the r_i for fixed n, hence attains a minimum. A short calculation shows that the result is least, for three points, when the second is half-way between the others. Hence the minimum for n points is not attained unless they are equally spaced. Therefore it is attained when they are equally spaced. The spacing is easily found to be given by $r_1 = 1/2n, r_2 = 1/2n + 1/n, r_3 = 1/2n + 2/n, \cdots, r_n = 1 - 1/2n$. To construct an infinite set, we bisect the intervals in the set just described to obtain a new set of the same type, and so on.

From time to time nD is as large as $9/32$ in this construction, and we are led to the following:

Query 3. What enumerable set $\{r_n\}$ minimizes $\overline{\lim}\, nD(E, r_i, n)$, and what is the minimum value?

The foregoing result suggests that

$$\underline{\lim}\, nD(E, r_i, n)$$

can be used as a measure of the efficiency with which E is approximated by $\{r_i\}$.

Remark 6. If $\underline{\lim}\, nD(E, r_i, n) = 1/4$, then $\{r_i\}$ is uniformly distributed; but if $\underline{\lim}\, nD(E, r_i, n) > 1/4$, the set need not be uniformly distributed.

If $\{r_i\}$ is not uniformly distributed there is a subinterval (α, β) of $[0, 1]$ which does not obtain its proper quota of points; $N(\alpha, \beta) \leqq \theta n(\beta - \alpha)$, say, with $\theta < 1$. The integral is least when the points in (α, β) are equally spaced, and those in the remainder of $(0, 1)$ are also equally spaced. Even then we find that $\lim nD > 1/4$, as $n \to \infty$, and the result follows. The second part is proved by the same calculation.

If $\{r_n\}$ is dense we have $\lim D(E, r_i, n) = 0$, as is obvious. It is a curious fact that a uniformly distributed set is no better than a dense set:

Remark 7. If $\lim d_n = 0$, there is a uniformly distributed set (depending on $\{d_n\}$) such that $D(E, r_i, n) > d_n$ for all large n.

Without loss of generality we may suppose $\{d_n\}$ decreasing. Take $r_1 = r_2 = \cdots = r_n = 1/2$ with n so determined that $d_n < 1/12$. Then take $r_{n+1} = \cdots = r_m = 1/4, 1/2, 3/4$ so that the numbers of points at these three positions are equal within ± 1 for large m. This enumeration should take due account of the n points at $1/2$. We choose m so that $d_m < 1/28$. Similarly, distribute points

$r_{m+1} \cdots r_p$ at 1/8, 2/8, 3/8, $\cdots$, as uniformly as possible, with due regard to those already present. We choose p large, and also such that $d_p < 1/60$. The process yields the desired set. By a slight modification one can avoid coincident points.

Note added in proof: According to Maurice Sion, a result of Morse and Beasley shows that the answer to Query 1 is affirmative; and a result of Besicovitch and Sierpínsky shows that the answer to Query 2 is negative, granted the continuum hypothesis.

A NOTE ON THE CIRCLE THEOREM IN HYDRODYNAMICS

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Milne-Thomson (see Proc. Cambridge Philos. Soc., vol. 36, 1940) found a certain function of a complex variable and showed that it had all the properties required for it to be the perturbation complex potential due to a circular cylinder placed at the origin in an infinite liquid in irrotational motion in two dimensions. The following is a method of obtaining the perturbation velocity potential in the above problem from simpler considerations.

Let $\phi(x, y)$ be the potential of a liquid which is in irrotational motion before the cylinder, $r = a$, is inserted. Then $\phi(x, y)$ is a harmonic function whose only singularities are outside $r = a$. Hence ϕ has an expansion given by $\phi = \sum_0^\infty r^n C_n$ valid in $r \leqq a$, where $C_n = p_n \cos n\theta + q_n \sin n\theta$, using (r, θ) as the plane polar co-ordinates. Hence

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = \sum_0^\infty n a^{n-1} C_n.$$

The inverse point of (r, θ) with regard to the circle $r = a$ is (R, θ), where $Rr = a^2$. If $\psi(x, y)$ is the perturbation potential due to the cylinder, then ψ must satisfy the following conditions:

(a) ψ is a harmonic function in two dimensions, regular at every point on and outside of $r = a$.

(b) $\psi = 0$ at $r = \infty$.

(c) $\dfrac{\partial \psi}{\partial r} = -\dfrac{\partial \phi}{\partial r}$ on $r = a$.

Thus ψ may be expanded in the form, $\psi = \sum_0^\infty r^{-n} C_n'$, where $C_n' = p_n' \cos n\theta + q_n' \sin n\theta$.

Condition (c) gives $\sum_0^\infty -n a^{-(n+1)} C_n' = \sum_0^\infty n a^{n-1} C_n$ for all θ in $0 \leqq \theta \leqq \pi$. So $p_n' = a^{2n} p_n$, $q_n' = a^{2n} q_n$, and $C_n' = a^{2n} C_n$. Hence $\psi(x, y) = \sum_0^\infty r^{-n} a^{2n} C_n = \sum_0^\infty R^n C_n = \phi(X, Y)$, where (X, Y) is the inverse point of (x, y), *i.e.*,

$$X = \frac{a^2 x}{r^2}, \qquad Y = \frac{a^2 y}{r^2}.$$

I wish to thank Dr. R. Mohanty for drawing my attention to the subject of this note.

CLASSROOM NOTES

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ON AN APPLICATION OF THE MEAN VALUE THEOREM

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In the study of the calculus, a number of important theorems are usually developed, but unfortunately their importance is not fully appreciated by beginning students. A useful application of the mean value theorem of the differential calculus is described below. The author has occasionally utilized this material with some success in indicating the importance and utility of this familiar theorem. Naturally all of the results in this paper are particular cases of Jensen's inequality, but the elementary way in which they may be derived apparently does not appear in the literature.

We establish first a lemma which is a direct consequence of the mean value theorem.

LEMMA. *If $a \leqq b \leqq c$, if $f'(x)$ is monotone increasing on the open interval $a < x < c$, and if $f(x)$ is continuous on the closed interval $a \leqq x \leqq c$, then*

$$(b - a)f(c) + (c - b)f(a) \geqq (c - a)f(b).$$

The inequality is reversed if $f'(x)$ is monotone decreasing. If further f' is strictly monotone and a, b, c are all distinct, equality is excluded.

Proof: Under the hypothesis the mean value theorem is valid so that we may write:

$$\frac{f(c) - f(b)}{c - b} = f'(\beta) \geqq f'(\alpha) = \frac{f(b) - f(a)}{b - a}.$$

Upon performing the indicated algebra, the desired result follows.

It is surprising that such an elementary device should prove rather powerful. To indicate its versatility we consider several examples. Let $f(x) = \ln (1+x)$. Then f' is monotone decreasing for $x \geqq 0$. Choose $\alpha > 0$, and let 0, α/n, α/m with $0 < m < n$ correspond to a, b, c of our lemma. Then

$$\frac{\alpha}{n} \ln \left(1 + \frac{\alpha}{m}\right) < \frac{\alpha}{m} \ln \left(1 + \frac{\alpha}{n}\right),$$

and we secure the familiar inequality:

$$(1 + \alpha/m)^m < (1 + \alpha/n)^n, \qquad m < n.$$

As a second example choose $f(x) = a^x$. Then f' is monotone increasing provided a is positive and different from unity. Let 0, $r-1$, and r be chosen as the points with $r > 1$. Then we obtain

$$(r - 1)a^r + 1 > ra^{r-1}.$$

Letting $a = 1/x$ we obtain the more familiar relationship:

$$x^r - 1 > r(x - 1).$$

It is easy to show that these inequalities are reversed if $0 < r < 1$ but unchanged if $r < 0$.

For a third example, let $f(x) = |x|^{1+\alpha}$, and let $\pm a$, $(x+a)/2$, and x be the numbers where $|x| \neq |a|$. Applying the lemma, we have:

$$(x \mp a)\,|\,(x \pm a)/2\,|^{1+\alpha} < [(x \mp a)/2]\,|\,x\,|^{1+\alpha} + [(x \mp a)/2]\,|\,a\,|^{1+\alpha},$$

or simplifying,

$$|\,x \pm a\,|^{1+\alpha} < 2^\alpha(|\,x\,|^{1+\alpha} + |\,a\,|^{1+\alpha}).$$

From symmetry it is unnecessary to consider the cases $x < a$ and $x > a$ separately. In passing we remark that this inequality can be used to show that if moments of order $1+\alpha$ exist, then moments of order $1+\alpha$ about the mean exist.

As a final example we apply our method to Liapounoff's famous moment inequality. Letting μ_x represent the absolute moment of order x, we seek to show that $\mu_b^{c-a} \leqq \mu_c^{b-a} \cdot \mu_a^{c-b}$. Choose $f(x) = \ln(\sum p_i x_i^x)$, $p_i > 0$, $x_i > 0$. Then

$$f''(x) = [\sum p_i x_i^x \sum p_i x_i^x \ln^2 x_i - (\sum p_i x_i^x \ln x_i)^2]/(\sum p_i x_i^x)^2.$$

Choosing $a_i = \sqrt{p_i} x_i^{x/2}$ and $b_i = \sqrt{p_i} x_i^{x/2} \ln x_i$, we find that $f'' > 0$ by the Cauchy-Schwartz inequality provided the x_i's are not all equal. Letting a, b, c be an increasing sequence of numbers, we therefore obtain

$$(b - a) \ln \mu_c + (c - b) \ln \mu_a \geqq (c - a) \ln \mu_b,$$

with equality holding only when all the x's are equal or when equality holds between two or more of the letters. Taking exponentials of both sides we have therefore proved the theorem for the case that X is a random variable assuming the value x_i with probability p_i. The theorem can now be extended to the continuous case as in the reference.

Reference

Uspensky, J. V., Introduction to Mathematical Probability, p. 265.

UNIFORMLY CONTINUOUS LINEAR SETS

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In this note, E will always denote a linear set of points. For E both closed and bounded (compact), it is well known that every continuous, real-valued $f(x)$ defined on E is uniformly continuous on E. The converse of this theorem is not true. For if E is the set of all positive integers, every single-valued, real function $f(x)$ on E is uniformly continuous, but E is not compact.

DEFINITION 1: *A linear set E will be termed a U.C. set if every real-valued continuous function $f(x)$ on E is uniformly continuous on E.*

It is the purpose of this note to characterize U.C. sets.

THEOREM 1. *Every U.C. set is a closed set.*

Proof: Let E be a U.C. set. If E is not closed, there is a point c which is a limit point for E, but which does not belong to E. Then, without loss of generality, there exists a sequence of points p_i in E such that $p_i < p_{i+1}$ and $\lim p_i = c$.

Define:

$$f(x) \equiv \begin{cases} 1 & \text{for } x = p_{2i-1} \\ 0 & \text{for } x = p_{2i} \end{cases} \qquad i = 1, 2, \cdots$$

and linear between p_i and p_{i+1}, and let $f(x) = 1$ for $x > c$, and 1 for $x \leqq p_1$. It is easy to verify that $f(x)$ is continuous on E, but not uniformly continuous on E, contrary to E being a U.C. set.

DEFINITION 2. *A set E is termed uniformly isolated if there exists an $\epsilon > 0$, such that for every $x \neq y$ in E, $|x-y| \geqq \epsilon$.*

THEOREM 2. *Every set E expressible in the form $E_1 + E_2$, where E_1 is compact and E_2 is uniformly isolated, is a U.C. set.*

Proof: Let $E = E_1 + E_2$, where E_1 is compact and E_2 is uniformly isolated. Let $A = E_2 - E_1$. Then A is uniformly isolated and hence closed, and $E = E_1 + A$. Let $f(x)$ be continuous on E. Then $f(x)$ is uniformly continuous on both E_1 and A. But since A is closed, and E_1 is compact, and A and E_1 are disjoint, A and E_1 are a positive distance apart. It is now clear that $f(x)$ is uniformly continuous on $E_1 + A$ and hence uniformly continuous on E. Therefore E is a U.C. set. Conversely, we have

THEOREM 3. *Every U.C. set E is expressible in the form $E_1 + E_2$, where E_1 is compact and E_2 is uniformly isolated.*

Proof: Let E be any U.C. set. Let $I_j \equiv \{x \mid -j \leqq x \leqq j\}$, $j = 1, 2, \cdots$. Let $E_+ \equiv \{x \mid x \epsilon E \text{ and } x \geqq 0\}$; and $E_- \equiv \{x \mid x \epsilon E \text{ and } x \leqq 0\}$. There exists a j' such that $E_+ - I_{j'}$ is uniformly isolated. For if $E_+ - I_j$ is uniformly isolated for no j, we can construct a sequence of points p_i in E such that $\lim p_i = +\infty$ and $p_i < p_{i+1}$ and

$p_{2i}-p_{2i-1}<1/i$. Then, as in Theorem 1, we can construct a continuous function $f(x)$ on E that is not uniformly continuous, contradicting E being a U.C. set.

Similarly, there is a j'' such that $E_{-}-I_{j''}$ is uniformly isolated. Now, let J be any integer greater than both j' and j''. By Theorem 1, E is closed and therefore $E\cdot I_J$ is closed and bounded, and hence compact. It follows from above that $E-I_J$ is uniformly isolated and $E=E\cdot I_J+(E-I_J)$. This proves the theorem.

The sum of two U.C. sets may not be a U.C. set. For if E_1 is the set $\{1, 2, \cdots, n, \cdots\}$ and E_2 is the set $\{1-1, 2-\frac{1}{2}, \cdots, n-1/n, \cdots\}$, then E_1 and E_2 are uniformly isolated and hence U.C. sets. However, E_1+E_2 is not a U.C. set as may be easily seen.

THEOREM 4. *Every closed subset of a U.C. set is a U.C. set.*

Proof: Let E be a U.C. set, and E_* a closed subset of it. By Theorem 3, $E=E_1+E_2$, where E_1 is compact and E_2 is uniformly isolated. Then $E_*=E_*\cdot E_1+E_*\cdot E_2$ and $E_*\cdot E_1$ is compact since E_* is closed, and $E_*\cdot E_2$ is uniformly isolated. Therefore, E_* is a U.C.set by Theorem 2.

COROLLARY. *The intersection of any family of U.C. sets is a U.C. set.*

Proof: This follows from Theorem 1 and Theorem 4.

BOUNDEDNESS OF A CONTINUOUS FUNCTION

M. D. MARCUS, University of British Columbia

Let f be a continuous real valued function and let M be a compact subset of the domain of f. It is classical that f is bounded on M and assumes its bounds. However, the following elementary proof using the Heine-Borel property may be of interest.

Let n be a positive integer and $I_n=(-n+f(x_0),\ n+f(x_0))$ for some $x_0\in M$. Then $\bigcup_{n=1}^{\infty}f^{-1}(I_n)$ is clearly an open covering of M. Select a finite subcovering

$$\bigcup_{j=1}^{K} f^{-1}(I_{n_j}) = f^{-1}(I_{\max n_j}) \supset M.$$

Then $-\max n_j\leqq f(x)-f(x_0)\leqq\max n_j$ and $f(x)$ is bounded. Now let $m=\text{lub}_{x\in M}f(x)$ and $J_n=(-\infty,\ m-1/n)$. Then if m is not assumed by f, $\bigcup_{n=1}^{\infty}f^{-1}(J_n)$ is an open covering of M and again select a finite subcovering

$$\bigcup_{j=1}^{K} f^{-1}(J_{n_j}) = f^{-1}(J_{\max n_j}) \supset M.$$

Then $f(x)\leqq m-1/\max n_j$, in contradiction to the definition of m.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1181. *Proposed by F. L. Wolf, Carleton College*

Show that if an integer has all its prime factors greater than 5 then it divides some integer whose decimal representation is a string of 1's.

E 1182. *Proposed by I. A. Barnett, University of Cincinnati*

If $d_k = u^k - x^k - y^k - z^k$, for $k = 1, 2, 3, \cdots$, establish the identity

$$d_{n+2} - (x + y + z)d_{n+1} + (xy + yz + zx)d_n - (xyz)d_{n-1} = u^{n-1}(u - x)(u - y)(u - z).$$

E 1183. *Proposed by F. J. Duarte, Caracas, Venezuela*

Prove that $\sum_{n=1}^{\infty}(n-1)/n! = 1$.

E 1184. *Proposed by Viktors Linis, University of Ottawa*

Let $f(x)$ be differentiable at $x = a$ and let $f(a) \neq 0$. Evaluate

$$\lim_{n\to\infty} [f(a + 1/n)/f(a)]^n.$$

E 1185. *Proposed by Frederic Cunningham, Jr., University of New Hampshire*

Prove that every infinite sequence of real numbers contains an infinite monotone subsequence.

SOLUTIONS

The Paint Paradox

E 1151 [1955, 121]. *Proposed by C. S. Ogilvy, Hamilton College*

Resolve the following paradox. The area between the curve $y = 1/x$ and the x-axis, to the right of the line $x = 1$, is infinite. Yet the volume generated by rotating this area about the x-axis is π. Thus it would require an infinite quantity of paint to cover the area; yet the volume, which completely contains and sur-

rounds the area, could be filled with π cubic units of paint!

Editorial Note. The general conclusions arrived at seem to be the following:

(1) If paint is molecular, then the paradox is explained by the fact that one can neither paint the area nor fill the volume after y becomes less than the molecular dimensions.

(2) If paint is continuous, then the paradox is explained by the fact that one needs but a finite amount of paint either to paint the surface or to fill the volume.

(3) If paint is mathematical, so that the paint on a painted surface has no thickness, then the paradox is explained by the fact that one cannot compare an area with a volume.

Contributions by R. V. Andree's Calculus Class, Julian Braun, M. L. Clinnick, Richard Courter, Hüseyin Demir, Fred Discepoli, J. M. Elkin, B. A. Fusaro, N. G. Gunderson, R. R. Gutzman, Vern Hoggatt, A. R. Hyde, E. S. Keeping, M. S. Klamkin, D. C. B. Marsh, J. V. Pennington, C. F. Pinzka, Anatol Rapoport, L. A. Ringenberg, Max Rosenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., Nina Schub, W. L. Shepherd, Lawrence Shepp, D. D. Strebe, D. R. Sudborough, J. A. Tierney, W. R. Van Voorhis, R. M. Warten, A. F. Wilson, R. J. Wisner, and the proposer.

Inconsistency of a Certain Postulate Set

E 1152 [1955, 122]. *Proposed by J. J. Bowers, Student, Wesleyan University*

If, in the postulates of a field, the distributivity of multiplication over addition is replaced by distributivity of addition over multiplication, is the resulting set of postulates consistent?

I. *Solution by C. J. Vanderlin, University of Wisconsin.* We shall denote by a *Bowers field*, or a *B-field*, a system $(F, +, \cdot)$ consisting of a non-empty set F together with two binary operations, $+$ and $\cdot$, such that:

(i) $(F, +)$ is an abelian group, with identity 0.
(ii) $(F^*, \cdot)$ is a group, with identity 1, where F^* is the set $F-0$.
(iii) $ab=ba$ for all $a, b \epsilon F$.
(iv) $a+bc=(a+b)(a+c)$ for all $a, b, c \epsilon F$.

As is customary, we shall denote the inverse of an element $a \epsilon F$ with respect to $+$ by $-a$, and the inverse of an element $b \epsilon F^*$ with respect to $\cdot$ by b^{-1}.

Lemma 1: Let $(F, +, \cdot)$ be a B-field. If $a \epsilon F$ and $a+1 \neq 0$, then $a=0$.

We have, by (iv),

$$a + 1 = a + 1 \cdot 1 = (a + 1)(a + 1).$$

Since $a+1 \epsilon F$ and $a+1 \neq 0$, then $a+1 \epsilon F^*$ and $(a+1)^{-1}$ exists. Therefore

$$(a + 1)^{-1}(a + 1) = (a + 1)^{-1}(a + 1)(a + 1),$$

or

$$1 = 1 \cdot (a + 1) = a + 1,$$

whence

$$1 - 1 = (a + 1) - 1 = a + (1 - 1),$$

or

$$0 = a.$$

Lemma 2: If $(F, +, \cdot)$ is a B-field, then the group $(F, +)$ consists of the elements 0 and 1 with $0+0=1+1=0$, $0+1=1+0=1$.

If $a \epsilon F^*$, then $a \neq 0$; so, by Lemma 1, $a+1=0$. But $1 \epsilon F^*$; so $1+1=0$. Therefore $a=1$, and 1 is the only element in F^*, and F consists of just the elements 0 and 1. Finally, $0+0=0$, $0+1=1+0=1$, since 0 is the identity of $(F, +)$.

Theorem: There exist no B-fields.

Let $(F, +, \cdot)$ be a B-field. Then

$$\begin{aligned} 1 + 1 \cdot 0 &= (1 + 1)(1 + 0) \quad \text{(by (iv))} \\ &= 0 \cdot 1 \quad \text{(by Lemma 2)} \\ &= 1 \cdot 0, \quad \text{(by (iii))} \end{aligned}$$

whence

$$1 = 0,$$

which is absurd. Therefore, there can exist no B-field.

It follows that the above postulate set is inconsistent.

Remark: If one allows the trivial system $(F, +, \cdot)$ with $F = \{0\}$, then this system satisfies the postulates for a B-field if (ii) is thought of as being vacuously satisfied. However, as is customary in field theory, we have assumed that F^* is a bona-fide group, that is, that it consists of a non-empty set.

II. *Addendum by C. N. Campopiano, Sperry Gyroscope Co., Great Neck, L. I.* Weakening the above postulate set for a B-field by deleting the commutivity postulate (iii), we find that the system $(F, +, \cdot)$, where $F = \{0, 1\}$ and $0+0=1+1=0$, $0+1=1+0=1$, $0 \cdot 0=0 \cdot 1=0$, $1 \cdot 1=1 \cdot 0=1$, satisfies the weakened set. Therefore, although the original set is inconsistent, the weakened set is not.

Also solved by W. J. Blundon, G. U. Brauer, Richard Courter, A. S. Gregory, J. D. Haggard, Douglas Holdridge, D. C. B. Marsh, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, H. M. Starr, D. R. Sudborough, and R. J. Wisner.

Editorial Note: It was impossible to check most of these solutions because the solver failed to indicate just what postulate set for a field was being employed. For example, many solvers used the fact that for all a, $a \cdot 0 = 0$. In the postulate

set for a field used above, this is a theorem whose proof depends upon the discarded distributivity law. As a matter of fact, the problem itself is at fault from this point of view; the proposer should have stated the postulate set for a field that he wished altered.

Trisection of Angles Arbitrarily Close to a Given Angle

E 1153 [1955, 122]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For any angle θ, show that arbitrarily small constructible angles ϕ exist such that $(\theta-\phi)$ can be trisected.

Solution by C. S. Ogilvy, Hamilton College. Let $\phi_n=\theta/4^n$, $n=1, 2, \cdots$. For all n, ϕ_n is certainly constructible. Obviously n can be selected so that $\phi_n<\epsilon$ for any ϵ. But

$$(\theta - \phi_n)/3 = \phi_1 + \phi_2 + \cdots + \phi_n.$$

Also solved by N. G. Gunderson, A. R. Hyde, J. V. Pennington, C. M. Sandwick, Sr., M. R. Spiegel, and the proposer.

A Property of the Regular Heptagon

E 1154 [1955, 122]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The distance from the midpoint of side AB of a regular convex heptagon $ABCDEFG$, inscribed in a circle, to the midpoint of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle.

I. *Solution by Leon Bankoff, Los Angeles, Calif.* Let d be the required distance, R the circumradius, and let $\theta=\pi/7$. By the cosine law,

$$d^2 = R^2(1/4 + \cos^2\theta - \cos\theta\cos 2\theta).$$

Since $\sin 3\theta=\sin 4\theta$, it follows that

$$3\sin\theta - 4\sin^3\theta = 4\sin\theta\cos\theta\cos 2\theta,$$

which reduces to

$$\cos^2\theta = \cos\theta\cos 2\theta = 1/4.$$

Hence $d=R\sqrt{2}/2$.

II. *Solution by Hüseyin Demir, Zonguldak, Turkey.* Let the vertices F, G, A, $\cdots$, E be the affixes of the 7th roots 1, ϵ, ϵ^2, $\cdots$, ϵ^6 of unity. Then the midpoints U, V of AB and the concerned radius correspond to $u=(\epsilon^2+\epsilon^3)/2$ and $v=-\frac{1}{2}$, whence

$$\begin{aligned} UV^2 &= (u - v)(\bar{u} - \bar{v}) = (1 + \epsilon^2 + \epsilon^3)(1 + \epsilon^5 + \epsilon^4)/4 \\ &= (2 + 1 + \epsilon + \cdots + \epsilon^6)/4 = 1/2, \end{aligned}$$

and the property is established.

Extending our method to diagonals, we may state the following: *The midpoints of the sides of the hexagon ABGDCEA are equidistant from the point V, the common distance being half the side of the inscribed square.*

Also solved by W. B. Carver, A. L. Epstein, Vern Hoggatt, A. R. Hyde, Edgar Karst, Josef Langr, D. C. B. Marsh, Walter Penney, J. V. Pennington, L. A. Ringenberg, C. M. Sandwick, Sr., E. P. Starke, Chih-yi Wang, and the proposer.

The Proposer mentioned the following additional properties of the regular heptagon $ABCDEFG$. Let O be the center of the heptagon, W the midpoint of OF, M the point diametrically opposite F, U the midpoint of AB, V the midpoint of OM, and J the point on UB produced such that $UJ = UM$. Then:

(1) UW is equal to the diagonal of the square constructed on an apothem of the heptagon as a side.

(2) UJ is equal to the diagonal of the square constructed on half the side of the inscribed equilateral triangle.

(3) UV is tangent to the circle through U, V, W.

One cannot help but wonder if these properties, and the property of the problem, are just remarkable geometric accidents, or are they special cases of more general theorems involving, perhaps, other regular polygons. Karst did some work on the regular inscribed nonagon $ABCDEFGHI$. Here, if U is the midpoint of side AB and V is the midpoint of the radius perpendicular to and cutting side BC, it can be shown that $\sphericalangle OUV = 30°$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4653. *Proposed by Albert Wilansky, Lehigh University*

Consider the three statements:

(a) $\sum U_n(x)$ converges uniformly, (b) $\sum |U_n(x)|$ converges,

(c) $\sum |U_n(x)|$ converges uniformly.

Consideration of $U_n(x) = (-x)^n/n$ shows that if we require (a) and (b) to

hold on an open interval (in this case $0<x<1$) we cannot deduce (c).

Does (c) follow if (a) and (b) hold on a closed interval?

4654. *Proposed by Harley Flanders, University of California at Berkeley*

Let $X=\{1, 2, 3, \cdots\}$ be the set of whole numbers, S the class of all subsets of X. Prove that there is a subclass T of S such that (a) if $Y_1, \cdots, Y_n$ are arbitrarily chosen distinct members of T, then there is at least one integer x which belongs to exactly an odd number of the Y_j, and (b) if Y is a non-empty member of S, then there are distinct sets $Y_1, \cdots, Y_n$ in T such that Y is the set of all integers x which belong to exactly an odd number of the Y_j.

4655. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Solve the difference equation

$$F(n+1) = F(n) - 2(n+1)\operatorname{sg}\{F(n) + 3n - n^2\}, \qquad F(0) = 0,$$

where $\operatorname{sg}(x)=1, 0, -1$ according as $x>0$, $x=0$, $x<0$.

4656. *Proposed by H. S. Shapiro, New York University*

(1) If α is a root of unity and its real part is an algebraic integer, then $\alpha^4=1$.

(2) Show that there exists an algebraic integer of absolute value 1 which is not a root of unity.

4657. *Proposed by Peter Swerling, Los Angeles, California*

Consider sets of real numbers which are subsets of the closed unit interval, which is denoted by I.

(1) Show that there exists no (Lebesgue) measurable set $S\subset I$ with the properties (i) $m(S)=M>0$ and (ii) there exists a number Δ, $0<\Delta<1$, such that for any interval $J\subset I$, $m(S\cap J)\leqq\Delta m(J)$. Here m denotes Lebesgue measure.

(1′) (Corollary) Given any measurable set $S\subset I$ such that $m(S)>0$ and given any number Δ, $0<\Delta<1$, there exists an interval $J\subset I$ such that $m(S\cap J)>\Delta m(J)$.

(1″) (Corollary) There is no measurable set $S\subset I$ with $m(S)=M>0$ such that S is "uniformly distributed" in I, in the sense that for every interval $J\subset I$, $m(S\cap J)=Mm(J)$.

(2) Show that statement (1′) cannot be improved in the following sense: one cannot say that for every measurable set $S\subset I$ with $m(S)>0$, there must exist a non-degenerate interval $J\subset I$ such that $m(S\cap J)=m(J)$.

(3) Given $S\subset I$; $m(S)>0$; and given Δ, $0<\Delta<1$, prove that S can be represented as the union of two disjoint sets: $S=S_1\cup S_2$ where (i) $m(S_1)=0$ and (ii) S_2 can be covered by a denumerable collection of disjoint non-degenerate intervals $\{I_n\}$: $S_2\subset\bigcup_n I_n$ such that for each I_n, $m(S\cap I_n)>\Delta m(I_n)$.

SOLUTIONS

Centroids of Parallel Convex Sets

4586 [1954, 264]. *Proposed by Robert Steinberg and F. A. Valentine, University of California at Los Angeles*

Consider a bounded closed convex set S_0 in the Euclidean plane. Let S_r be the parallel convex set to S_0, that is, the set of all points whose minimum distance from S_0 is at most r. Designate the centroid of S_r by g_r.

Show that the set of points $g_r (0 \leqq r < \infty)$ is a point or an arc of an hyperbola (possibly degenerate) lying in S_0. Does g_∞ have a simple geometric relationship with S_0?

Solution by the Proposers. Select a rectangular coordinate system (x_1, x_2), let $\overline{S}_0$ be the boundary of S_0, and let ϕ be the angle which the outward normal $N(\phi)$ to $\overline{S}_0$ makes with the x_1-axis. A point p in $S_r - S_0$ corresponds to a pair (r_1, ϕ), where r_1 is the distance of p from $N(\phi) \cdot \overline{S}_0$. Then the area of the ring-like set $S_r - S_0$, denoted by A_r is

$$A_r = \int_0^{2\pi} \int_{\rho_0}^{\rho_0 + r} \rho d\rho d\phi = \int_0^{2\pi} \int_0^r (r_1 + \rho_0) dr_1 d\phi = \pi r^2 + L_0 r,$$

where $\rho_0 = \rho_0(\phi)$ is the radius of curvature of $\overline{S}_0$ at $N(\phi) \cdot \overline{S}_0$, L_0 is the length of $\overline{S}_0$, and $\rho_0 d\phi = ds$ on $\overline{S}_0$.

In a corresponding manner, the moment of $S_r - S_0$ about the x_2-axis can be shown to be

$$\int_0^{2\pi} \int_{\rho_0}^{\rho_0 + r} (x_1 + r_1 \cos\phi) \rho d\rho d\phi = \int_0^{2\pi} \int_0^r (x_1 + r_1 \cos\phi)(\rho_0 + r_1) dr_1 d\phi$$

$$= r \int_0^{L_0} x_1 ds + \frac{r^2}{2} \int_0^{2\pi} x_1 d\phi,$$

where (x_1, x_2) is the point $N(\phi) \cdot \overline{S}_0$. A corresponding result holds for the x_1-axis. Hence, if we let A_0 designate the area of S_0, then the x_i-coordinate of g_r is

$$\bar{x}_i = \frac{\iint_{S_0} x_i dA + r \int_0^{L_0} x_i ds + \frac{r^2}{2} \int_0^{2\pi} x_i d\phi}{A_0 + rL_0 + \pi r^2}, \qquad (i = 1, 2). \tag{1}$$

If $\overline{S}_0$ is a circle, clearly g_r is always its center. Otherwise, the isoperimetric inequality $(L_0^2 - 4\pi A_0 > 0)$ implies that equations (1), with r as parameter, represent an hyperbola, a straight line or a point. Moreover, equations (1) imply

$$\min_{x_i \in S_0} x_i \leqq \bar{x}_i \leqq \max_{x_i \in S_0} x_i, \qquad (i = 1, 2).$$

Hence $g_r \in S_0$.

If S_0 has interior points, by choosing (0, 0) on $\overline{S}_0$, and by choosing x_1 along a normal to $\overline{S}_0$, it is easy to see that g_r is in the interior of S_0. Equations (1) also yield the fact

$$\lim_{r\to\infty} \bar{x}_i = \frac{1}{2\pi}\int_0^{2\pi} x_i d\phi.$$

This describes $\lim g_r$ in terms of S_0. It is interesting to observe that the set of all g_r is a point if and only if, relative to g_r as origin, we have

$$\iint_{S_0} x_i dA = 0, \qquad \int_0^{L_0} x_i ds = 0, \qquad \int_0^{2\pi} x_i d\phi = 0. \tag{2}$$

The question naturally arises: What are the sets S_0 satisfying equations (2)?

An Impossible Property of Solutions of a Differential Equation

4591 [1954, 350]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(z)$ be analytic. For $a=1$, the equation $f'(ax)+f(x)=0$ $(-\infty<x<\infty)$ has the property that its solution, $f(x)=e^{-x}$, approaches zero as $x\to\infty$. Is this true for any real $a>1$?

Solution by Paul Erdös, University of Notre Dame, and Melvin Henriksen, Purdue University. An elementary computation with power series shows that the only non-zero solutions of $f(x)+f'(ax)=0$ are necessarily analytic and are given by

$$f(x) = A\sum_{n=0}^{\infty} (-1)^n (n!)^{-1} a^{-n(n-1)/2} x^n, \qquad A \neq 0.$$

It is easily seen that if $a>1$, $f(x)$ is an entire function of order zero. By a theorem of Wiman (see Titchmarsh, *Theory of Functions*, p. 274), there exists a sequence x_n, approaching infinity, such that $f(x_n)\to\infty$. (Such a sequence could be explicitly constructed in this special case.)

Note: The trivial exception $f(x)=0$ is not explicitly excluded in the statement of the problem.

Also solved by I. N. Baker, D. S. Greenstein, Edgar Reich, O. E. Stanaitis, and the Proposer.

Dirichlet Series

4592 [1954, 350]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the sum

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log r}{r}.$$

I. *Solution by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland.* Write

$$f(n) = \sum_{r=1}^{n} \frac{\log r}{r} - \frac{1}{2}\log^2 n,$$

then from Cauchy's integral test it follows that $f(n)$ tends to a limit as $n \to \infty$: hence

$$f(2n) - f(n) \to 0, \qquad n \to \infty.$$

Now, since $\log 2r = \log 2 + \log r$,

$$\sum_{r=1}^{n} \frac{\log r}{r} = 2\sum_{r=1}^{n} \frac{\log 2r}{2r} - \log 2 \sum_{r=1}^{n} \frac{1}{r};$$

hence

$$f(2n) - f(n) = \sum_{r=1}^{2n} \frac{(-1)^{r+1}\log r}{r} + \log 2\left(\sum_{r=1}^{n} \frac{1}{r} - \log n\right) - \frac{1}{2}\log^2 2.$$

Thus, letting n tend to infinity, we have

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log r}{r} = \frac{1}{2}\log^2 2 - \gamma \log 2,$$

where γ denotes Euler's constant.

II. *Solution by M. R. Spiegel, Rensselaer Polytechnic Institute.* Choose $a > 0$ and consider

$$\begin{aligned}\int_0^{\infty} \frac{\log x}{e^{ax}+1}\,dx &= \sum_{r=1}^{\infty} (-1)^{r+1} \int_0^{\infty} e^{-rax} \log x\, dx \\ &= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{ra} \int_0^{\infty} e^{-u} \log (u/ra)\,du \\ &= \left\{\int_0^{\infty} e^{-u} \log u\, du\right\} \left\{\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{ra}\right\} - \sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log ra}{ra}.\end{aligned}$$

Since

$$\int_0^{\infty} e^{-u} \log u\, du = \Gamma'(1) = -\gamma,$$

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{ra} = \frac{\log 2}{a},$$

$$\int_0^{\infty} \frac{\log x}{e^{ax}+1}\,dx = -\frac{1}{2a}\log 2 \log 2a^2,$$

(see solution to problem 4394, [1951, 705]) we obtain

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} \log ra}{ra} = \frac{1}{2a} \log 2 \log 2a^2 - \frac{\gamma \log 2}{a}.$$

The present problem is the special case $a = 1$.

Also solved by P. T. Bateman, Ranko Bojanić, W. E. Briggs, L. Carlitz, S. Chowla, Oscar Goldman, S. W. Golomb, Norman Greenspan, J. W. Lawson, D. E. Russell, O. E. Stanaitis (two solutions), Arnold Walfisz, C. B. Walton, Chih-yi Wang, R. E. Wild, and the Proposer.

Editorial Note. In a note on *The Power Series Coefficients of* $\zeta(s)$, Briggs and Chowla obtain, among other results, formulas for the coefficients A_n in the expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n (s-1)^n.$$

See this MONTHLY, May 1955, pp. 323–325.

Residue Systems

4593 [1954, 427]. *Proposed by S. W. Golomb, Harvard University*

Let $0, a_1, a_2, \cdots, a_{p-1}$ be any complete residue system modulo the odd prime p. Show that

$$0, a_1, 2a_2, \cdots, (p-1)a_{p-1} \tag{1}$$

is never a complete residue system modulo p.

Show further that if any non-zero residue r is specified, the a_i's can be so chosen that every residue except r occurs in (1). What relation will the residue s occurring twice bear to r?

Solution by D. H. and Emma Lehmer, Berkeley, California. It is clear that the zero in the problem may be dispensed with by replacing "complete system of residues" by "the set S of numbers less than p." Now if corresponding elements of two permutations of S are multiplied together and reduced modulo p the resulting set is not a permutation of S since, by Wilson's theorem, the product of the members of S is congruent (mod m) to -1.

For the second part of the problem we may take the set defined by

$$a_k \equiv \begin{cases} 1 & \text{if } k = p - r, \\ 1 + r\bar{k} & \text{otherwise,} \end{cases} \qquad k\bar{k} \equiv 1 \pmod{p}.$$

Since $r \not\equiv 0$, it is clear that this set of a's is a complete set of residues prime to p. Moreover

$$ka_k \equiv \begin{cases} p - r & \text{if } k = p - r \\ k + r & \text{otherwise} \end{cases} \pmod{p}$$

is also such a set except that r is missing and $s=p-r$ occurs twice, once for $k=p-r$ and again for $k=p-2r$.

Also solved by Kurt Bing, W. J. Blundon, L. Carlitz, S. H. Gould, D. S. Greenstein, Virginia S. Hanly, B. A. Hausmann, W. H. Jones, D. C. B. Marsh, R. R. Phelps, and the Proposer.

Zero of a Complex Function

4594 [1954, 427]. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z)=z$ whenever $|z|=1$, show that $f(z)$ has at least one zero in $|z|<1$.

I. *Solution by A. R. Hyde, West Hartford, Connecticut.* If $-1<x_1<1$, from the definition of $f(z)\equiv u+iv$, $u=x_1$ at a pair of points on the circle $|z|=1$, symmetric with respect to the axis of reals. Because of the continuity of $f(z)$ there is at least one continuous path which connects these points and lies within the circle and along which $u=x_1$. Correspondingly for $v=y_1$ (where the point pair on the circle are symmetric with respect to the axis of imaginaries). The paths $u=x_1$ and $v=y_1$ must intersect (within the circle) provided $|z_1|\equiv|x_1+iy_1|<1$, and at such point or points of intersection $f(z_1)=x_1+iy_1=z_1$. Thus at least once within $|z|=1$, $f(z)$ assumes every value assumed by z, including $f(z)=0$, which occurs at the intersection of $u=0$ (connecting $z=\pm i$) and $v=0$ (connecting $z=\pm 1$).

II. *Solution by S. K. B. Stein, University of California at Davis.* If $f(z)=0$ has no solution in $|z|<1$, then the continuous map $z\to f(z)/|f(z)|$ provides a retraction of the disk into its boundary. For any dimension this is known to be impossible. See Hurewicz and Wallman, *Dimension Theory*, Proposition B, p. 40.

III. *Solution by Albert Wilansky, Lehigh University.* Let

$$g(z)=\begin{cases} z & \text{for } |z|\geqq 1,\\ f(z) & \text{for } |z|<1.\end{cases}$$

Then g is continuous for all z and $g(z)/z\to 1$ as $z\to\infty$. By the result of Problem 4475 [1953, 271], g has a zero. It has none on or outside the unit circle, hence has one inside.

Also solved by J. L. Brown, Jr., H. J. Cohen, A. J. Goldman, C. D. Gorman, R. H. Kasriel, G. J. Kleinhesselink, Norman Miller, L. A. Ringenberg, Azriel Rosenfeld, Michael Skalskyj, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent to E. P. Vance, Oberlin College, Oberlin, Ohio

An Introduction to Statistics. By C. E. Clark, New York, John Wiley and Sons, 1953. x+266, pages. $4.25.

This text is written for students with a minimal mathematical background, presupposing less than a mastery of high school algebra. Yet, unlike many such texts, the concern here is primarily with statistical inference and only those descriptive statistics most frequently used in statistical inference are introduced. Thus measures of central tendency and dispersion other than the mean and standard deviation are omitted entirely, as are index numbers, time series analysis, measures of skewness and kurtosis, methods of machine computation, *etc.* Whether or not one believes, as this reviewer does, that the inclusion of such topics in an introductory course tends only to obscure the more important ideas of estimation and tests of hypotheses, it is clear that the author's intention is to introduce statistical inference to the mathematically immature student, and it is with respect to this intention that the book must be evaluated.

The first four chapters are devoted to an introduction to the concept of statistical inference, permutations and combinations, probability, and frequency and probability distributions. The exposition in this section, and throughout the book, is clear and well organized and is supplemented by a carefully chosen set of exercises, half of which are solved for the reader in an appendix.

In the chapter on probability a distinction is made between three kinds of probability—empirical, statistical, and *a priori*—and an attempt is made to maintain this distinction throughout the text, but it seems to this reviewer that the use of a single definition would have been preferable. What the author defines as empirical probability could better be called relative frequency or sample proportion, the term used in later chapters in which the problem of the estimation of the unknown parameter of a binomial population is considered. The concept of *a priori* probability, as defined by the author, is applicable only to idealized games of chance which can be analyzed in terms of a finite set of equally likely outcomes. We are left, then, with statistical probability, defined as a subjective expectation of the frequency of occurrence of an event in future trials. This reviewer prefers a single definition similar to that used by Feller in his *Probability Theory and its Applications*, with the author's empirical probability treated as a sample estimate, his *a priori* probability treated as a special case of "fair" coins and dice, and his statistical probability treated as descriptive of the kinds of probability statements made in problems of estimation by confidence limits and tests of hypotheses.

The text follows a pattern of heuristic discussion, formal statement of a theorem, and formal proof whenever possible within the limits set by the low

level of mathematical training required of the reader. When a rigorous proof is not presented, a theorem is made plausible by unusually clear informal arguments. This mathematical format serves well to emphasize the important results obtained: a section is summarized in a brief and usually very careful statement in the form of a theorem. However, this technique of emphasis is occasionally misused. For example, the author chooses to define a frequency or a grouped probability distribution to be a normal distribution if the probabilities of a pair of tables hold. These tables give areas under the normal curve between ordinates symmetrically placed about the mean. (The normal curve is pictured only once, above a table of normal distribution areas at the end of the book.) A frequency distribution is defined to be "roughly normal" if the probabilities of these tables hold approximately. Then the following is stated as a formal theorem (4.25.3): "Almost all the numbers of a roughly normal frequency distribution differ from the mean by less than 3 times the standard deviation." It seems to this reviewer that to make this a formal theorem is to give a spurious validity to a statement which is in fact almost devoid of content.

Problems of statistical inference are discussed in four chapters titled The Reliability of Sample Means and Proportions, The Significance of the Difference between Two Sample Means or Percentages, The Analysis of Variance, and Inferences from Chi-Square. Primary emphasis is placed upon the estimation of population parameters by means of confidence limits. Since the concept of the null hypothesis is first mentioned in the chapter on the chi square test, tests of hypotheses concerning the mean are limited to statements of the risk (one minus the confidence) with which one can infer from a random sample that the population mean is less than (or greater than) a particular value. This treatment of tests of hypotheses makes difficult the introduction of the concept of the power of a test, and in fact the possibility of error of the second kind, acceptance of a false hypothesis, is never treated. Tests of hypotheses concerning a binomial population are also handled by the use of confidence limits based upon the normal approximation to the binomial. Here a serious error is introduced in the text by the estimate of the standard error using the sample proportion rather than the population proportion under the null hypothesis. However this problem is handled correctly and clearly in several of the exercises.

The analysis of variance is treated briefly in a short chapter which describes only the case of a single variable of classification. The exposition is clear and the usual notational complications are kept to a pleasant minimum, but it seems unfortunate even in so brief a treatment that no justification is given for using as a statistic the ratio of the between means and within samples estimates of variance and that no mention is made of the conditions under which the analysis of variance is valid.

After a short chapter on the chi-square test, the author turns in the last chapter from statistical inference back to the description and analysis of empirical data by means of correlation and linear regression. That these should be handled only as descriptive statistics in a text largely devoted to statistical inference

seems regrettable. In fact, problems of correct inference certainly are raised by statements such as that on page 204: "A value of $r(x, y)$ between .6 and .7 or $-.6$ and $-.7$ indicates a significant connection." Either this only defines significant connection or else an inference to a population parameter is being made, with no mention of sample size, upon which the confidence with which an inference can be made depends.

J. A. DUDMAN
Reed College

Partial Differential Equations in Engineering Problems. By K. S. Miller. New York, Prentice Hall, Inc., 1953. viii+254 pages. $4.75.

According to the author's introduction, this book is directed to upper classmen or beginning graduate engineering students. It deals with the elementary aspects of the theory of partial differential equations of mathematical physics without pretense of rigor, and presupposes an elementary knowledge of ordinary differential equations.

Chapter I contains the derivation of some of the equations in "typical engineering fashion." It is concluded with a three page list of equations. In Chapter II Fourier series are discussed. Except for partial statement (without proof) of two finer points (Section 26 on Points of Discontinuity, and Section 40 on Mathematical Conditions on Fourier Series Expansions) this discussion consists essentially of derivation of formulae for Fourier coefficients, and of computation of various examples, spread over 40 pages. A natural continuation of the topic is found in Chapter V on Legendre, Bessel, and Mathiew functions. The defining equations for these functions are obtained in the process of applying the method of separation of variables to equations involving the Laplacian in spherical, cylindrical, and elliptical coordinates. Orthogonality relations are obtained, and the theory of representation of functions as linear superpositions over a complete orthonormal set is explained. Chapter V ends with a brief survey of the Sturm-Liouville Theory. More material of preparatory character is found in Chapter IV on the Fourier Integral.

The discussion of partial differential equations in this book is of rather limited scope. Chapter III on Separation of Variables contains the discussion of four boundary value or mixed boundary value-initial value problems for second order linear equations with constant coefficients in two variables leading to Fourier series expansions; an example of a fourth order equation in two variables (transverse vibrations of a beam); and finally, a discussion of the rectangular vibrating membrane without consideration of the possibility of degeneracy of eigenvalues. Some nonhomogeneous equations are also considered and the method of reduction of a problem with nonhomogeneous equation to a problem with homogeneous equation but with nonhomogeneous boundary conditions is explained. In Chapter IV there are two sections in which Fourier transforms are used to solve the Cauchy problem for the heat and the wave equation. In Chapter V there are three sections devoted to solutions by the method of separation of variables

in which generalized Fourier series are used. Lastly, in the concluding Chapter VI the reduction to normal form of second order equations in two variables with constant coefficients is carried out.

The book is written in a clear, unhurried style, and there is no doubt that the author wanted the reader to understand everything without effort. Yet one cannot escape the feeling that this book constitutes a literal printing of unpolished lecture notes. Uniformity of presentation is lacking. For example, in setting up problems, in some cases units are used (as if they were of importance) while in other cases they are not used, and certain definitions are entirely forgotten (compare Sections 2, 4, 5, and 7). Similar comments may be made concerning what is considered as the prerequisite for the reader. For example, it is assumed that the reader may not know the mean value theorem of the differential calculus, and half of page 2 is given to the proof assuming the knowledge of Taylor series. On page 187 it is shown that $\lim_{x\to\infty} x^a e^{-x} = 0$, while on page 190 the author assumes that the method of Frobenius is known. Matter having no bearing on the subject is sometimes considered. For example, there is in Chapter V a four page section on the gamma function, but the function itself is used only in connection with expansions for Bessel functions (pages 202 to 205) where it appears with integral arguments. Likewise, in connection with the study of Bessel functions, functions of negative index are considered.

The author made an attempt to be specific and precise, but often misses the point. In setting up problems, he carefully lists all the assumptions (although no distinction is made between assumptions of physical nature and mathematical simplifications). Then on page 4 we find:

"Assumption 2. The deflections u are small compared with the length L of the string.

Assumption 3. The slope at any point of the deflected string is small compared to unit."

as if the former were not implied by the latter. Incidentally, this particular list of assumptions is followed by a comment that they are reasonable and consistent and then one reads a few lines below: "An example of a pair of inconsistent assumptions would be to assume that the displacements u are large compared with the length of the string, but that the horizontal displacements are small."

One usually expects a textbook on a mathematical subject to be mathematically palatable. Yet we find in this one a whole spectrum of sins from loose language to outright errors. In Chapter I the use of such undefined phrases as "elementary," "infinitesimal," "in the first approximation" is commonplace. On page 29 a too crude differentiation of integrals is punished by obtaining the Laplace equation in place of the Poisson equation. From Chapter II the student may get the impression that only periodic functions can be expanded in Fourier series, while in Chapter IV he is told that Fourier transforms are considered to take care of the aperiodic case. In Section 49 the notion "nonhomogeneous boundary conditions" does not agree with the accepted meaning of this phrase.

In Chapter V, the study of the Legendre equation is started on the wrong foot (by deriving the indicial equation in a regular case as the Legendre equation is at $x=0$) and is ended likewise.

Our main criticism concerns, however, the presentation of material. One gets the feeling that by remaining on a too elementary level, the issues in question are only confused. Would it not be better to start the Chapter on Fourier series with a discussion of trigonometric approximations (as in Section 27), and after a few applications to Fourier series, to proceed to generalized Fourier series? This approach, while clarifying the purpose of the subject, would considerably shorten the presentation of preparatory material, and would allow the author to expand on the main subject. As it is, the selection of material is unbalanced and is certainly inadequate for a student who may not have a chance to return to the subject.

I. I. KOLODNER
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New York University

Mathematical Thinking in the Social Sciences. Edited by Paul F. Lazarsfeld. Glencoe, Illinois, The Free Press, 1954. 444 pages. $10.00.

Decision Processes. Edited by R. M. Thrall, C. H. Coombs and R. L. Davis. New York, John Wiley and Sons, Inc., 1954. viii+332 pages. $5.00.

Problems in economics, psychology, or sociology do not yet generally leap to mind when applications of mathematics are under discussion. Nevertheless, there is a growing awareness, among both mathematicians and social scientists, that mathematical thinking and methods are important in these fields. The volumes under review should make significant contributions to this growth. Each is a collection of articles concerned with the development of mathematical theories and techniques, with applications of this mathematics to social science problems, or with methodological questions of how mathematical models may best be used by social scientists.

Mathematical Thinking in the Social Sciences contains 8 long papers in addition to an introductory essay by the editor. T. W. Anderson defines an attitude as a choice from among a set of responses to a question and then describes a Markov chain probability model for analyzing changes in people's attitudes over time. This analysis is then applied to data obtained in a panel survey of voters. Nicholas Rashevsky, with his work in mathematical biology as background, derives differential equations for studying the behavior of large social groups, especially imitative and mob behavior. He also develops a theory of social status distributions in human populations. The third paper, by James C. Coleman, is an analysis of four of Rashevsky's social behavior models. Eliminating mathematical details, he compares these models in order to explain the interplay between their formal mathematical and their empirical sociological content.

The nature and role of the probability concept in social science are reviewed by Jacob Marschak. He discusses subjective probabilities, the statistical character of descriptive social science, and the use of probability models in forming policy decisions. The problems of aggregation and identification in economics are outlined and relations of these with the sociologist's latent structure and the psychologist's factor analysis are indicated.

In the first of two articles, Louis Guttman interprets the characteristic vectors of a certain matrix as so-called principal components of scalable social attitudes. This theory of scale analysis is a remarkable example of productive interplay between mathematics and social science. The author refers elsewhere for the mathematical details and limits himself to the psychological theory and its relation to the mathematical formalism. In his second article, Guttman uses quantitative data from mental testing as illustrative material and develops a new general theory for linking test scores and the factors studied by these tests. This radex theory extends the classical factor analysis approaches of Spearman and Thurstone, using multivariate statistical analysis as its main mathematical tool.

The central problem considered in the paper by Paul F. Lazarsfeld is how to analyze a set of qualitative responses to test questions or observed items of behavior (the so-called manifest data) in order to yield measurements of underlying characteristics (the so-called latent structure of the subject group). In this presentation of latent structure analysis we find discussions of the notion of probability as used in the theory, of response patterns, and of the "accounting equations", by which inferences are made from the manifest data to the latent structure.

The final contribution, by Herbert A. Simon, expresses the need for translating into the language of mathematics some of the central laws in social-psychological theory and for considering many complementary models for phenomena to be studied. Two main examples of model building are given, one a study of motivation and learning processes and the other an examination of equilibrium states in a system of social interaction.

Decision Processes is a volume on the proceedings of a 1952 summer seminar which brought together a group of mathematicians, psychologists, economists, and philosophers. After two introductory articles, one a discussion of the papers to follow and their interrelationships, the other an exposition of the nature of mathematical models with special emphasis on models in measurement theory, the remaining 17 papers are divided into four subject matter groups I. Individual and Social Choice, II. Learning Theory, III. Theory and Applications of Utility, IV. Experimental Studies. In general, all of these papers, ranging in character from pure mathematics to experiments in group dynamics, deal with one aspect or another of research into procedures and rationale for making decisions or into how decisions are actually made. All told, one obtains that variety, in both subject matter and approach, necessary to present a fair sample of the recent work in the field of decision problems.

Questions may be raised as to the advisability of presenting, as both volumes do, so uneven a product in terms of prerequisites needed for successful reading. We believe this method of presentation is advisable. It enables the editors not only to point out the rich possibilities of mathematical social science, but at the same time to emphasize the fact that mathematician and social scientist need to work together to continue to explore these possibilities.

SAMUEL GOLDBERG
Oberlin College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

OFFICERS OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 20–24, 1955, at Pennsylvania State University. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor Haim Reingold, Illinois Institute of Technology; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Professor C. O. Oakley, Haverford College. Dr. R. S. Burington, Bureau of Ordnance, Navy Department, and Dr. C. V. Newsom, Executive Vice-Chancellor, New York University, will continue to serve as Directors. Dr. Burington was elected Council Member of ASEE. The next annual meeting of the ASEE Mathematics Division will be held in June, 1956, at Iowa State College. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

PERSONAL ITEMS

Professor C. A. Hutchinson of the University of Colorado represented the Association at the dedication of the United States Air Force Academy in Denver, Colorado, on July 11, 1955.

Assistant Professor B. F. Bryant of Vanderbilt University is on leave of absence for 1955–1956 on a Ford Foundation Faculty Fellowship and is at Princeton University.

Georgetown University announces: Dr. Anne Scheerer of Washington University has been appointed to an assistant professorship; Mr. L. N. Zaccaro of Temple University has been appointed to an instructorship.

Kansas State College reports the following: Dean R. W. Babcock of the School of Arts and Sciences retired from the deanship July 1, 1955, after twenty-five years of such service but will continue as Professor of Mathematics; Professor R. G. Sanger has been appointed Acting Dean; Mr. J. J. Harris and Miss Dorothy Powell have been appointed to instructorships.

Los Angeles City College announces that the fifth annual William B. Orange Mathematics Prize Competition for Los Angeles high school students was held May 13, 1955, with 171 students from 38 schools participating.

Assistant Professor N. E. Albrecht of Hamline University has a position as a mathematician with Remington-Rand Corporation, St. Paul, Minnesota.

Associate Professor C. E. Amos of the University of Toledo has been promoted to a professorship.

Dr. E. L. Arnoff, recently a research associate at Case Institute of Technology, has been appointed to an assistant professorship at the Institute.

Mr. F. S. Badger, dean of students at Alliance College, has been appointed to an associate professorship at Allegheny College.

Dr. W. E. Barnes, previously head of Ballistics and Statistical Theory Branch, U. S. Naval Proving Ground, Dahlgren, Virginia, has a position as a mathematical consultant at General Electric Company, Schenectady, New York.

Associate Professor Max Beberman of Florida State University has been appointed Associate Professor of Education at the University of Illinois.

Dr. Barbara J. Beechler of the State University of Iowa has been appointed to an assistant professorship at Wilson College, Chambersburg, Pennsylvania.

Associate Professor J. H. Blau of Antioch College is Acting Chairman of the Department of Mathematics until September 1956.

Dr. S. R. Bodner, formerly a research assistant at the Polytechnic Institute of Brooklyn, has a position as a senior mathematician at Republic Aviation Corporation, Farmingdale, New York.

Mr. H. A. Bott, previously a student at Illinois Institute of Technology, is employed as a research engineer at Minneapolis-Honeywell, Chicago, Illinois.

Dr. J. W. Brace of the University of Maryland has been promoted to an assistant professorship.

Assistant Professor F. R. Brown of Illinois State Normal University has been promoted to an associate professorship.

Professor W. F. Cheney, Jr., of the University of Connecticut has retired from his position at the University and is Head of the Department of Mathematics of Hillyer College.

Associate Professor R. N. Cobb of Worcester Polytechnic Institute has been promoted to a professorship.

Mr. S. H. Cohn, formerly an instructor at Fournier Institute of Technology, is now in the Aerodynamics Computing Centre, Avro Aircraft Limited, Malton, Ontario, Canada.

Associate Professor V. F. Cowling of the University of Kentucky has been promoted to the position of Professor of Mathematics and Astronomy.

Mr. D. W. Crowe, previously a research assistant at the University of Michigan, has been appointed to an instructorship at the University of British Columbia.

Mr. R. S. Dick, recently a student at Queens College, New York, is a University Fellow in Mathematical Statistics at Columbia University.

Assistant Professor M. H. M. Esser of Georgia Institute of Technology has been appointed to an instructorship at the University of Maryland.

Mr. A. R. Friedenheit, formerly a mathematician at Douglas Aircraft Corporation, Santa Monica, California, is employed now as a mathematician-programmer at the Burroughs Corporation, Paoli, Pennsylvania.

Dr. R. F. Gabriel of Rutgers University has been promoted to an assistant professorship.

Assistant Professor David Gans of New York University has been promoted to an associate professorship.

Dr. E. F. Gillette of Harpur College has been appointed to an assistant professorship at Union College.

Assistant Professor Samuel Goldberg of Oberlin College has been promoted to an associate professorship.

Dr. J. K. Goldhaber of Cornell University has been appointed to an assistant professorship at Washington University.

Associate Professor A. W. Goodman of the University of Kentucky has been promoted to a professorship.

Dr. R. T. Gregory, formerly an assistant at the University of Illinois, has been appointed to an assistant professorship at the University of California, Santa Barbara College.

Assistant Professor Dilla Hall of Southern Illinois University has been promoted to an associate professorship.

Associate Professor J. R. Hammond of the United States Naval Academy has been promoted to a professorship.

Mr. C. L. Hassell, Jr., previously a supervisor of the Mathematical Operations Section, Sandia Corporation, Albuquerque, New Mexico, has a position as a research mathematician at the Continental Oil Company, Ponca City, Oklahoma.

Assistant Professor F. V. Higgins of Fenn College has been promoted to an associate professorship.

Mr. H. L. Hunzeker, instructor at DePauw University, has been promoted to an assistant professorship.

Dr. D. A. Kearns of the University of Maine has been promoted to an assistant professorship.

Mr. J. D. E. Konhauser, recently an instructor at Pennsylvania State University, has a position as a mathematician at Haller, Raymond and Brown, Inc., State College, Pennsylvania.

Mr. E. L. Kretschmar, Jr., formerly a mathematics teacher at Pasco High

School, Dade City, Florida, has been appointed Assistant Principal of Zephyrhills Public Schools, Florida.

Dr. E. B. Leach of Massachusetts Institute of Technology has been appointed to an assistant professorship at Case Institute of Technology.

Professor Jean Leray of the Institute for Advanced Study and College de France lectured in the Department of Mathematics of the Institute of Technology, University of Minnesota, on topics in partial differential equations, September 9–25, 1955.

Mr. R. L. Liboff, formerly a physicist at the Army Chemical Center, Maryland, is now a junior research physicist at Stevens Institute of Technology.

Mr. K. L. Loewen, previously an instructor at Freeman Junior College, South Dakota, has been appointed to an assistantship at Pennsylvania State University.

Mr. J. N. Mangnall, associate mathematician at Cornell Aeronautical Laboratory, Buffalo, New York, has been appointed to an instructorship at the University of Buffalo.

Mr. G. J. Marks, recently a test engineer with Curtiss-Wright Corporation, Woodridge, New Jersey, is employed now as a technical engineer in the Jet Engine Department of General Electric Company, Evendale, Ohio.

Mr. R. L. May, previously a technical assistant at Socony-Vacuum Oil Company, New York City, is now a naval architect for California Texas Oil Company, New York City.

Professor C. T. McCormick of Illinois State Normal University has been appointed Head of the Department of Mathematics.

Assistant Professor S. W. McInnis of the University of Florida has been promoted to an associate professorship.

Assistant Professor E. K. McLachlan of Baylor University has been promoted to an associate professorship.

Mr. V. A. Miculka, chairman of the Department of Mathematics of Frank Phillips College, has been appointed to an instructorship at Texas Western College.

Dr. Knox Millsaps, chief of the Applied Mathematics Research Branch of the Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio, is Visiting Professor at Massachusetts Institute of Technology for the academic year 1955–1956.

Dr. Marian A. Moore, chairman of the Department of Mathematics of Glenbrook High School, Northbrook, Illinois, has been appointed to an assistant professorship at Southern Illinois University.

Mr. W. E. Moore, previously chief of the Data Reduction Section of Redstone Arsenal, has a position as a senior computer engineer in the Engineering Division of Republic Aviation Corporation, Farmingdale, New York.

Dr. Z. I. Mossesson, chief actuarial assistant at Prudential Insurance Company of America, Newark, New Jersey, has been promoted to the position of associate actuarial director.

Mr. Herbert Nadler, recently in military service, has been appointed to the position of actuary with the Fire Department Pension Fund, New York City.

Mr. Hiram Paley, formerly a student at the University of Rochester, is now a National Science Foundation Fellow at the University of Wisconsin.

Dr. B. J. Pearson, previously a teaching fellow at the University of Texas, has been appointed to an instructorship at Illinois Institute of Technology.

Dr. L. L. Philipson of the Lockheed Aircraft Corporation has accepted a position as research engineer with the Hughes Research and Development Laboratories, Culver City, California.

Miss Lois J. Roper, formerly instructor at Trenton Junior College and Senior High School, is teaching at St. Joseph Junior College, Missouri.

Associate Professor E. H. Rothe of the University of Michigan has been promoted to a professorship.

Mr. M. J. Sendrow of the Department of Defense, Washington, D. C., has a position as an engineer at the R. C. A. Engineering Products Division, Camden, New Jersey.

Assistant Professor R. L. Shively of Western Reserve University has been appointed Oppenheim Professor and Head of the Department of Mathematics of Manchester College.

Assistant Professor E. C. Smith, Jr., of the University of Utah has a position as Applied Science Representative with the I. B. M. Corporation, San Francisco, California.

Dr. O. K. Smith of Princeton University has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. J. K. Sterrett has accepted a position with the Office of Naval Research, Operations Research Division, Navy Department, Washington, D. C.

Professor W. R. Van Voorhis of Fenn College has been appointed Chairman of the Department of Mathematics.

Mr. W. D. Ward, previously a student at the University of Michigan, is now a student actuary for Aetna Life Insurance Company, Hartford, Connecticut.

Mr. C. B. H. Watson, formerly a student at the University of Toronto, is now an actuarial student at Canada Life Assurance Company, Toronto, Ontario, Canada.

Associate Professor H. P. Wirth of the City College of the City of New York has been promoted to a professorship.

Assistant Professor Rhoda Manning Wood of Oregon State College has been promoted to an associate professorship.

Professor Emeritus N. R. Bryan of the University of Maine died on May 22, 1955. He was a member of the Association for thirty-five years.

Professor Emeritus Cora Strong of Woman's College, University of North Carolina, died on June 3, 1955. She was a member of the Association for thirty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 119 persons have been elected to membership by the Board of Governors on applications duly certified.

SMBAT ABIAN, M.S. (Chicago) Grad. Student, University of Cincinnati.

LUCILLE J. ALBERS, B.S. (Dayton) Data Processing Mathematician, Land-Air Inc., Wright Patterson Air Force Base, Ohio.

M. C. ALTSCHULER, Student, Yale University.

J. J. ANDREWS, B.A. (Hofstra) Teaching Asst., University of Georgia.

J. F. ANDRUS, B.S. (Charleston) Grad. Student, Emory University.

C. E. ANTLE, B.S. (Eastern Kentucky S.C.) Grad. Asst., University of Kentucky.

ZAMIR BAVEL, B.A. (Southern Illinois) Grad. Asst., Southern Illinois University.

W. H. BENSON, B.S. (U.S. Naval Acad.) Captain, United States Navy.

A. F. BOND, Jr., Student, West Virginia University.

R. V. BORCHERS, B.S. (Iowa S.C.) Cartographer, U.S. Forest Service, Alexandria, Va.

I. W. BOXER, M.A. (Calif. State Poly. C.) Instr., California State Polytechnic College.

C. B. BROWN, Student, Oklahoma Agricultural and Mechanical College.

DONALD BROWN, Student, Hofstra College.

R. T. BUMBY, Student, Massachusetts Institute of Technology.

F. A. CENEY, JR., Student, Southern Illinois University.

N. H. CLARK, Student, University of California at Los Angeles.

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K. L. CONRAD, M.S. (Iowa S.C.) Jr. Dev. Engr., Goodyear Aircraft Corp., Akron, O.

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W. C. COOK, Student, State College of Washington.

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R. T. DONNELL, M.A. (Vanderbilt) Department Head, Union University.

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E. S. EBY, Student, University of Illinois.

CATHERINE G. EHREN, Student, Cardinal Stritch College.

R. M. ELASHOFF, A.M. (Boston U.) Grad. Student, Boston University.

E. A. ENRIONE, Student, University of Miami.

HERBERT ERICSON, JR., Lieutenant, United States Air Force.

C. G. FAIN, B.A. (Tulsa) Electronic Calculator Programmer, Eglin Air Force Base, Fla.

R. B. FOSTER, JR., Student, University of the South.

NORMA M. FREY, B.S. (Nevada) Grad. Student, University of Colorado.

W. P. GANLEY, Student, University of Buffalo.

A. M. GARNER, JR., B.A. (A.&M.C. of Texas) 2/Lt., United States Army.

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WALTER GUBER, New York, New York.

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D. J. HENDERSON, Student, University of British Columbia.

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G. E. MARTIN, M.A. (N.Y.S.C. for Teachers, Albany) Batavia, New York.

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HALCYON E. MCNEIL, B.S. (Kansas S.C.) Carter Oil Company, Tulsa, Okla.

R. J. MCQUILLIN, Student, College of Puget Sound.

D. A. MERLINE, Student, University of South Carolina.

A. D. MILLER, Ph.D. (Iowa S.C.) Asso. Professor, Wisconsin State College.

VIVIAN MORGAN, Student, State College of Washington.

MISS MASAKO OBA, Student, University of California at Los Angeles.

M. A. OMAR, Student, Colorado School of Mines.

M. L. PATRICK, Student, Eastern Kentucky State College.

PAUL PAYETTE, Student, Ecole Polytechnique de Montréal, Université de Montréal.

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A. W. RANSOM, Student, University of Rochester.

W. K. RAWDON, Student, Yale University.

W. A. REES, M.A. (Texas) Asst. Professor, University of Houston.

BARNETT RICH, Ph.D. (Columbia) Chairman, Brooklyn Technical High School, Brooklyn, N. Y.

J. I. RICHARDS, Student, University of Minnesota.

GEORGE RICHMAN, Student, University of California at Los Angeles.

R. A. RIEGER, Asst. Personnel Manager, Comar Electric Co., Chicago, Ill.

R. B. RODMAN, Student, Reed College.

K. W. ROESSING, Student, West Virginia University.

I. S. RUBIN, M.S (N.Y.U.) Mathematician-Programmer, Remington Rand, New York, N. Y.

DAVID SACHS, Student, Illinois Institute of Technology.

R. L. SANDERS, B.S. in Min.E. (Missouri School of Mines) Geophysicist, Stanolind Oil and Gas Co., Tulsa, Okla.

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INEZ I. SAUSEN, A.B. (Marymount) Instr., Garden City Junior College, Kansas.

R. J. SCHEFFEL, Student, Hofstra College.

E. C. SCHLUTER, Student, Municipal University of Omaha.

C. E. SEABOLD, B.A. (Heidelberg) Mathematician, Wright-Patterson Air Force Base, Ohio.

RAYMOND SEICK, Student, College of Puget Sound.

BEN SILVER, S.B. (M.I.T.) Providence, R. I.

A. H. SIMON, Student, Cooper Union.

MRS. JANET M. SMITH, Student, Brown University.

O. P. STACKELBERG, Student, Massachusetts Institute of Technology.

J. R. STALLINGS, JR., Student, University of Arkansas.

J. C. STROUD, B.A. (Colorado C.) Mathematician, National Bureau of Standards, Boulder, Colo.

GEORGE SWARTFIGURE, Student, University of Buffalo.

D. F. TEMPLETON, Jr., Student, American University.

BARBARA J. THOMSON, Student, University of Oregon.

J. K. THURBER, Student, Brooklyn College.

C. W. TRAFTON, Undergraduate Asst., Iowa State College.

L. P. TUELL, Student, Northeastern State College.

R. J. TUKOVITS, Student, Hunter College of New York.

P. L. WARNSHUIS, JR., M.S. (Stanford) Mathematician, U. S. Naval Ordnance Test Station, Pasadena, Calif.

IMELDA WASINGER, Student, Marymount College.

J. W. WEIHE, Ph.D. (California, Berkeley) Staff Member, Sandia Corp., Albuquerque, N. M.

L. R. WELCH, B.S. (Illinois) Grad. Student, California Institute of Technology.

H. R. WESSON, B.S. (Birmingham-Southern) Grad. Asst., University of Alabama.

D. R. WILDER, B.A. (Oberlin) U. S. Naval Reserve.

BENJAMIN WILLIAMS, Student, Virginia State College.

R. G. WILLIAMS, M.S. (Illinois) Instr., Sullivan School, Washington, D. C.

L. L. WIMP, M.A. (Missouri) Lecturer, Southern Illinois University.

LEM WONG, Student, University of California at Los Angeles.

M. D. WOODARD, M.S. (Washington) Grad. Asst., University of Florida.

C. R. B. WRIGHT, Student, University of Nebraska.

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-first annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on April 24, 1955. Professor L. K. Jackson of the University of Nebraska presided.

There were 40 persons in attendance, including the following 26 members of the Association:

M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Camp, Morris Dansky, H. W. Doss, Jr., J. M. Earl, M. G. Gaba, Edwin Halfar, L. K. Jackson, M. L. Keedy, W. G. Leavitt, K. L. Loewen, E. J. Lowry, R. L. Moenter, T. A. Newton, Florence E. Pool, H. B. Ribeiro, H. L. Rice, Lulu L. Runge, E. C. Schluter, George Seifert, T. C. Smith, H. P. Thielman, J. F. Wampler.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor E. J. Lowry, Hastings College; Vice-Chairman, Professor L. K. Jackson, University of Nebraska; Secretary, Professor Edwin Halfar, University of Nebraska.

The following papers were presented:

1. *An infinite series analogue to the gamma function*, by Dr. T. A. Newton, University of Nebraska.

2. *An application of independence*, by Professor W. G. Leavitt, University of Nebraska.

The idea of independence of sets of functions may be used to give rather simple proofs of several theorems from elementary analysis. Suppose $\{f_i\}$ $(i=1, \cdots, n)$ is an independent set of analytic functions, which is also closed under differentiation (that is, $f_i' = \sum_{j=1}^{n} C_{ij}f_j$). Let $F(x) = \sum_i^n C_i f_i$ for an arbitrary set $\{C_i\}$ of constants. The following theorems may be proved:

THEOREM 1. *If a is a constant such that $e^{-ax} \neq \sum k_i f_i$ for any set $\{k_i\}$ of constants, then integration by parts furnishes a method of evaluating the integral $\int e^{ax}F(x)dx$.*

THEOREM 2. *If $\{a_i\}$ $(i=1, \cdots, k)$ is a set of constants none of which are characteristic values of the matrix $[C_{ij}]$ then the method of undetermined coefficients may be used to solve the differential equation $(D-a_1)\cdots(D-a_k)y=F(x)$.*

3. *An isoperimetric inequality*, by Professor M. A. Basoco, University of Nebraska.

Let $p(\theta)$ be the supporting function of a convex curve C in the plane. This function is characterized by the conditions that (1) it be periodic, period 2π, and (2) it be "sub-sine" in the sense of Pólya and J. W. Green. If $p(\theta)$ is sufficiently regular, the length L of C, the area F of the region bounded by C and the area E of the evolute of C may be expressed in terms of the Fourier coefficients of $p(\theta)$. Following methods initiated by A. Hurwitz, it is shown that the isoperimetric deficit $(L^2/4\pi) - F \leqq E/4$. A characterization of those convex curves for which equality holds is obtained.

4. *Generalizations of the concept of continuity*, by Professor H. P. Thielman, Iowa State College. (By invitation.)

Let f be a function on a neighborhood space X onto a subset of a neighborhood space Y. The neighborhoods in X are denoted by N_x, those in Y by M_y. The symbol $f^{-1}(M_y)$ indicates the subset of X which consists of those elements x of X for which $f(x)$ is an element of M_y. Let λ be a property of subsets S of X. The property λ is said to be an ascending set property if λ is such that if S has the property λ and T is any subset of X, then $S+T$ also has the property λ. In general, $f(a)$ is said to be λ-approached if for every $M_{f(a)}$ the set $F^{-1}(M_{f(a)})N_a$ has the property λ. The speaker showed that a number of well-known concepts such as continuity almost everywhere, partial continuity, neighborliness and many others are special types of λ-approaches. It was shown that a number of theorems on convergent sequences of continuous functions can be extended to sequences of λ-approach functions if λ is an ascending set property. Various examples of λ-approach functions were given. In particular, an example was given of a function which has the Darboux property (*i.e.*, the property of taking on all values between any two functional values) which is cliquish but is not neighborly. For the definition of these concepts the reader is referred to: Thielman, H. P., *Types of Functions*, this MONTHLY, vol. 60, 1953, pp. 156–161.

5. *A class of spherical hyperbolas*, by Professor O. C. Collins, University of Nebraska.

The locus of constant angle of the sphere has been explored for all combinations of the two parameters, subtended angle and subtending arc. This curve has been employed as a tool in the study of one of its own special cases, where it degenerated into a spherical hyperbola of a particular class, in which the minor axis subtends a right angle at all points on the curve. A formula was found for the angle subtended at any point on this curve by any chord perpendicular to the major axis. The angles subtended at the ends of any diameter were found to be either supplementary or equal according as that diameter does or does not intersect at the chord. The minimum angle was found to be subtended at either end of the minor axis.

6. *On the misuse of accuracy*, by Professor Robert M. Kozelka, University of Nebraska.

This inquiry into certain spurious, or merely misleading, uses of "significant" figures included examination of some possible causes and effects of such misuse and examples from advertising, teaching, and government documents.

7. *Engineers need competence in calculation*, by Professor T. C. Smith, University of Nebraska.

Design today implies dozens of variables that must be dealt with simultaneously. Engineers must set up and solve the corresponding simultaneous equations. Engineers should be taught efficient methods for solving such equations. To illustrate this point, the author explained a method of condensation for solving n simultaneous linear equations with n unknowns. This method has been used by his descriptive geometry class to verify the solution of certain space problems.

Furthermore, one need not hesitate to work with numbers having three or more digits if a calculating machine is used. The author predicted that in the not too distant future a fairly elaborate calculating machine will be a necessary tool for mathematicians and engineers.

8. *Academic background of college students at the University of Nebraska*, by Professor H. M. Cox, Director, Bureau of Instructional Research, University of Nebraska.

The patterns of courses taken in high school by entering freshmen at the University of Nebraska are significantly different in certain areas by sex and by choice of curriculum. They doubtless reflect, in part, the pattern of requirements for admission to the college or curriculum chosen by the students. The differing patterns of courses taken by student groups explain, in part, the differences in distributions of scores made by the students on placement examinations in mathematics, English, *etc.*

9. *Late developments in arithmogeometry*, by Mr. H. W. Becker, Omaha, Nebraska.

(A) It was proved that if one side of a Pythagorean triangle is an mth power not divisible by m, then neither of the other two sides can be an mth or any other power.

(B) 200 errata in Dickson's *History*, v. II, 1st edition, were classified in six categories. These include some noted by Alliston, Bell, Carmichael, Lehmer, Ward, and Dickson himself later.

(C) By five different methods of proof, Pythagorean kites, tetrahedra, cuboids, *etc.*, can have no general solution in polynomials, hence neither can the underlying equation "general quartic $= \square$." This liberates Diophantine Analysis from its most modern supersitition.

EDWIN HALFAR, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The fifteenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on April 8, 1955. Professor D. L. Webb, Chairman of the Section, presided.

There were 53 persons in attendance, including the following 34 members of the Association:

C. E. Aull, C. E. Buell, J. H. Butchart, Louis Child, R. B. Crouch, M. E. Drummond, Jr., G. E. Forsythe, J. H. Fountain, F. C. Gentry, R. F. Graesser, Elzie A. Greene, J. D. Hankins, P. W. Healy, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, M. H. Hoehn, Carl Holtom, T. L. Jordan, Jr., Max Kramer, R. B. Lyon, W. W. Mitchell, Jr., D. R. Morrison, E. A. Propes, E. J.

Purcell, B. D. Roberts, C. B. Rogers, Rafael Sanchez-Diaz, E. A. Voorhees, Earl Walden, D. L. Webb, R. L. Westhafer, H. H. Wicke, G. M. Wing.

The following officers were elected for the year 1955: Chairman, Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanical Arts; Vice-Chairman, Professor R. B. Lyon, Arizona State College at Tempe; Professor W. W. Mitchell, Jr., Phoenix College, continues for the second year of a four year term as Secretary-Treasurer. The traveling lectureships were abolished.

An invited address "Digital Computing at U.C.L.A." was given following the banquet by Dr. G. E. Forsythe of the University of California at Los Angeles. Dr. Forsythe also read a paper on "Computing Constrained Maxima with Lagrange Multipliers" during the morning session.

The following papers were presented:

1. *A new treatment for the parabola of surety*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

Professor Butchart showed that the envelope of the paths of projectiles issuing from a fixed point with a fixed initial velocity, which is a parabola with its focus at the point of departure, can be developed using only elementary physics, polar coordinates, and pure geometry. He noted the ease with which incidental properties can be proved such as the following: (a) the locus of the foci of the trajectories is a circle; (b) the directrices of the trajectories coincide with the tangent at the vertex of the envelope; (c) if angles of departure differ by 90°, the points of contact with the envelope are at ends of a focal chord.

2. *Cross ratio in n dimensions*, by Professor F. C. Gentry, University of New Mexico.

The cross ratio of two points a and b and two hyperplanes $(mx)=0$ and $(nx)=0$ in projective space of n dimensions was defined as $[(ma)(nb)]/[(mb)(na)]$. From this it was proved that the cross ratio of 4 hyperplanes is constant if they belong to a pencil and that the cross ratio of 4 points is constant if they are collinear. The definition was shown to be consistent with the usual definition for one dimensional space.

3. *Bi-geometric progressions*, by Dr. D. R. Morrison, Sandia Corporation, Albuquerque, New Mexico.

A progression is bi-geometric if there exist constants p and q, such that for every three consecutive terms, x, x', x'' of the progression, $x''=px'+qx$. Rules for geometric progressions presented in standard algebra texts are extended to bi-geometric progressions, using only elementary algebra and analytic geometry. Formulas are developed for the sum and the nth term and for interpolation. Applications are made to oscillating circuits, vibrating strings, pendula, and hanging cables, which would otherwise require differential equations, and to discrete problems to which differential equations are not applicable, such as compound interest with modified interest rates and annual population growth in symbiotic situations.

4. *Sextactic points of a plane curve*, by Professor Louis Child, New Mexico College of Agriculture and Mechanical Arts.

At the point P of a base curve Γ, at which the radius of curvature ρ and its first four derivatives $\rho_1, \rho_2, \rho_3, \rho_4$ with respect to the arc-length s exist and are finite, and $\rho \neq 0$, there exists a one-parameter family D_7 of seven-pointic osculating nodal cubics. Every member of D_7 is nondegenerate if $q \equiv 36\rho_1+9\rho^3\rho_3-9\rho\rho_1\rho_2+4\rho_1^3 \neq 0$. If $q=0$, then every member of D_7 is composite, consisting of the osculating conic and a line on P. The osculating conic is then sextactic; it is not sextactic when $q \neq 0$.

5. *On computing constrained maxima with Lagrange multipliers*, by Dr. G. E. Forsythe, University of California, Los Angeles.

Let R be the n-dimensional space of points x. Let G be the intersection of m smooth surfaces $g_r(x)=0$ $(r=1, \cdots, m)$. A computer wishes to locate a certain point $x_{\max}$ which locally maximizes $f(x)$ with respect to G. A classical method of Lagrange multipliers suggests defining $F(x, \lambda)=f(x)+\sum_{r=1}^{m}\lambda_r g_r(x)$, where $\lambda=(\lambda_1, \cdots, \lambda_m)$ are constants. One then seeks to adjust λ until the appropriate stationary value y of $F(x, \lambda)$ with respect to R lies in G. Then ordinarily $y=x_{\max}$.

It is shown by an example $(n=2, m=1)$ and a theorem that, *although $x_{\max}$ is a local maximum of $f(x)$ with respect to G, it may be only a saddle point of $F(x, \lambda)$ with respect to R.* When so, $x_{\max}$ could not be found by ordinary ascent methods applied to $F(x, \lambda)$ in R. A possible cure is to apply an ascent method to $f(x)$ within G.

6. *The splitting of monomial groups*, by Professor R. B. Crouch, New Mexico College of Agriculture and Mechanical Arts.

This paper will be published in the *Transactions of the American Mathematical Society.*

7. *A development of cardinals in "The Consistency of the Continuum Hypothesis,"* by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

In *The Consistency of the Continuum Hypothesis* the axiom of choice is used to construct the theory of powers within that of cardinals. The main purpose here is to develop cardinals without such an axiom. This work could be formalized with the use of the predicate calculus, but, as is done in the book, the proofs are presented rather informally.

8. *The mathematical history of 781 integral calculus students*, by Professor D. L. Webb, University of Arizona.

A summary of the courses and grades of 781 University of Arizona students who have completed integral calculus in the years from 1948 to 1954.

9. *Arizona mathematics contest*, by Professor W. W. Mitchell, Jr., Phoenix College.

The development, execution, and preliminary results of the first annual Arizona Mathematics Contest sponsored by the Arizona Association of Teachers of Mathematics were explained. Copies of the examination and contest rules were distributed and a short discussion period followed.

10. *The use of visual aids in mathematics courses*, by Professor P. W. Healy, University of New Mexico.

Professor Healy presented his views concerning the use of visual aids in mathematics courses. He pointed out that increased use of improved visual aids could be a partial answer to the dilemma facing us in the immediate future when there will be more students and fewer qualified teachers. His presentation was followed by some sound films designed to teach mathematical topics.

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Abilene, Texas, on April 22–23, 1955, sponsored jointly by Abilene Christian College, Hardin-Simmons University, and McMurry College. Sessions were held Friday afternoon and Saturday morning, Professor E. A. Hazelwood and Professor M. E. Mullings presiding.

By invitation, Professor W. L. Duren, Jr., of Tulane University delivered an address at the afternoon session. The title of his address was "A Course on Sets." At the banquet Professor Dorothy McCoy of Wayland Baptist College gave an illustrated lecture on her teaching experiences in Iraq.

There were 74 persons in attendance, including the following 50 members of the Association:

T. A. Abouhalkah, C. P. Benner, Ina M. Bramblett, H. E. Bray, J. E. Burnam, L. A. Colquitt, Don Cude, H. B. Curtis, Jr., W. L. Duren, Jr., B. T. Goldbeck, Jr., R. G. Green, R. E. Greenwood, Jr., W. T. Guy, Jr., Lena L. Hays, E. A. Hazelwood, E. R. Heineman, R. F. House, P.W.M. John, Guy Johnson, Jr., E. C. Klipple, J. W. Lindsay, C. W. Long, H. A. Luther, Dorothy McCoy, R. V. McGee, A. B. McLemore, R. M. McLeod, V. A. Miculka, P. D. Minton, V. H. Morrill, M. E. Mullings, C. J. Pipes, C. L. Riggs, Virginia B. Roberts, C. R. Sherer, D. P. Shore, W. C. Sikes, S. A. Sims, Sister Antonietta, D. W. Starr, J. R. Swaffield, Jennie L. Tate, W. W. Taylor, J. I. Tracey, F. E. Ulrich, R. S. Underwood, Patricia A. Ward, Mabel Williams, C. B. Wright, Martin Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor M. E. Mullings, Abilene Christian College; Vice-Chairman, Professor Don Cude, Southwest Texas State Teachers College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *A Mathieu integral transform*, by Professor W. T. Guy, Jr., The University of Texas.

A Mathieu integral transform was defined and three useful operational properties were given. Then, these properties were used to solve a differential equation and derive a relation between two special functions.

2. *A Taylor's series*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

Let λ be a complex number and let $f(z-z_0)$, where $f(0)=1$, be a Taylor's series in powers of $z-z_0$. It can be shown that there exists a Taylor's series in $z-z_0$ which can consistently be interpreted as $[f(z-z_0)]^\lambda$. Moreover, the coefficients of this latter series can be exhibited. In this discussion, however, there is presented only an example of non-obvious character. The proof used is independent of the general results mentioned above.

3. *On bounded families of analytic functions*, by Mr. Guy Johnson, Jr., The Rice Institute.

Let F denote a uniformly bounded family of functions analytic in a domain D. The function $G(z)=\sup_F |f(z)|$, called the modulus of F, is continuous and satisfies a maximum principle. If $G_1(z)$ is defined in the annulus $R_1<r<R_2$ by $G_1(z)=M(r)$ for $|z|=r$, and $M(r)$ is a convex function of $\log r$ then there exists a family F_1 for which $G_1(z)$ is the modulus.

4. *A course on sets*, by Professor W. L. Duren, Jr. Tulane University.

The Committee on the Undergraduate Program of MAA has been consulting with mathematicians, engineers, and scientists on the content of the early college courses in mathematics. Present first year courses fall into about six diverse types. This paper presents an appeal for simplifying these offerings into one course for normally prepared students. Such a course already exists in the form of graphs and elementary calculus, and numerous text books are available for it. This

elementary calculus should be devoted to the study of polynomial, rational, exponential, and logarithmic functions, postponing analytic trigonometry.

Early calculus is feasible if certain traditional but unnecessary prerequisites are eliminated. A major source of these unessential subjects is in the analytic geometry of the euclidean plane. A more general "graphs" geometry in the xy-plane is better. The graphs geometry admits straight lines, slopes, and area under a curve but not angle, circle, and arc length.

5. *Combinatorial relations and chromatic graphs*, by Professor R. E. Greenwood, Jr., The University of Texas.

The results presented in this talk have been published in the following paper: Greenwood, R. E. and Gleason, A. M., *Combinatorial relations and chromatic graphs*, (Canadian Journal of Mathematics, vol. 7, 1955, pp. 1–7).

6. *Birth and death processes*, by Mr. P. W. M. John, University of Oklahoma.

The probabilities of population sizes in a generalized birth and death process are given by the infinite set of differential equations.

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t).$$

Complete solutions are known for those processes in which λ_n and μ_n are either constant or may be written as $\lambda(t)n$ and $\mu(t)n$. The state of knowledge of estimators for the parameters λ and μ for stationary processes of the latter type is discussed.

7. *An extension of gap theorems to expansions in Gontcharoff polynomials*, by Mr. R. M. McLeod, The Rice Institute.

The classical gap theorems on Taylor's series are extended to series $\sum_{n=0}^{\infty} c_nP_n(z)$ where $\{P_n(z)\}$ is the sequence of Gontcharoff polynomials corresponding to a sequence $\{z_n\}$ satisfying

$$\limsup_{n\to\infty} |z_n|^{1/n} = a, \qquad a < 1.$$

The method is to show that the region of regularity of the difference

$$\sum_{n=0}^{\infty} c_n(P_n(z) - z^n)$$

includes the closed disk $|z| \leqq R$ in its interior, R being the radius of convergence of $\sum c_nz^n$.

8. *Related algebraic and geometric curiosities*, by Professor R. S. Underwood, Texas Technological College.

One way to get often predictable loci on a plane for equations in n variables is to use an inflexible coordinate system. This method turns up an algebraic curiosity—the analogue of the "point-ellipsoid,"—whose single solution is known here but hard to find otherwise. A second method solves two quadratics simultaneously by adjusting the plotting rule so that the locus of one equation is a line or curve. The other locus turns out to be the "geometric curiosity"—a closed area bounded by an ellipse, two parabolas, and a straight line.

9. *How to solve it*, by Professor J. I. Tracey, Texas Christian University.

Among the problems included in this paper are some for which the solution can be obtained readily by the use of the harmonic mean. Others which are suggested include the famous Morley Triangle and some of the historical problems from Archimedes' *Method*.

10. *A theorem on the roots of the derivative of a polynomial*, by Professor H. E. Bray, The Rice Institute.

A theorem of A. Cohn states that if $P(z)$ is a polynomial of degree $2k$, with k pairs of roots, each pair having the same argument, ϕ_j, and moduli $(r_j<R_j)$ such that $r_jR_j=1$, $[j=1, 2, \cdots k]$, then the derivative $P'(z)$ has exactly k roots outside the unit circle. The theorem is generalized; the conclusion holds if the moduli r_j, R_j, are such that:

$$r_j < 1 < R_j; \qquad a_j \geqq 1; \qquad h_j < 9a_j/(a_j^3 + 8) \quad \text{if} \quad 1 \leqq a_j \leqq 2,$$

where a_j is the arithmetic mean of r_j, R_j, and h_j is their harmonic mean, $j=1, 2, \cdots k$.

11. *Engineers and mathematicians*, by Mr. T. A. Abouhalkah, Railroad Commission of Texas.

Expert teachers who will inspire and encourage prospective engineers and mathematicians at high school and college levels are badly needed for the sake of sciences and of civilization. Successful mathematicians have shown that the laws of nature are designed mathematically, that the relation of the universe with what is inside of its horizons is based on sound mathematics, and that the concepts of physical and social sciences have been influenced by the comprehension and understanding of the "Queen of the Sciences."

Successful engineers are not interested only in methodology and technique, but consider mathematics as an "intellectual occupation." Hence, they are not deprived of the charm of the subject and they do not resort blindly to the using of formulae in solving their engineering probems.

12. *Current legislation affecting higher education*, by Professor E. A. Hazelwood, Texas Technological College.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-ninth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 7, 1955, at Duquesne University, Pittsburgh, Pennsylvania. Professor F. H. Steen, Chairman of the Section, presided at the morning and afternoon sessions.

There were 63 in attendance, including the following 42 members of the Association:

A. G. Anderson, Thomas Bauserman, J. O. Blumberg, W. G. Brady, A. M. Bryson, F. C. Calabrese, J. G. Christiano, A. B. Cunningham, D. A. De Felice, Esther S. Dunkelberger, R. D. Edwards, Ruth O'Donnell Goodman, D. S. Hoffman, B. P. Hoover, M. A. Hyman, F. E. Justis, J. C. Knipp, E. C. Kovacs, George Laush, Gloria A. Martin, R. G. McDermot, David Moskovitz, L. T. Moston, B. H. Mount, Jr., E. F. Myers, L. A. Ondis, II, Morris Ostrofsky, M. C. Palmer, P. A. Penzo, I. D. Peters, F. H. Steen, Andrew Sterrett, Jr., Alexander Strasser, E. A. Sturley, W. C. Styslinger, Jr., J. S. Taylor, Jean E. Teats, V. E. Thomas, L. O. Thompson, E. A. Whitman, P. M. Whitman, B. H. Youell, Jr.

At the business meeting, the following officers were elected for a term of two years: Chairman, Dean L. T. Moston, Waynesburg College; Secretary, Professor I. D. Peters, West Virginia University; Executive Council: Professor F. H. Steen, Allegheny College; Professor R. D. Edwards, University of Pittsburgh.

The following papers were presented:

1. *A plastic theory for pre-sintered powdered metals*, by Mr. E. M. Shoemaker, Carnegie Institute of Technology, introduced by the Chairman.

A mathematical model is constructed which is intended to characterize the mechanical behavior of a pre-sintered powdered metal. Suitable plastic potential surfaces are discussed and the corresponding stress-strain laws derived. Uniqueness theorems and minimum principles are established.

2. *An experiment in probability*, by Professor Emeritus E. A. Whitman, Carnegie Institute of Technology.

In this paper the author shows one arrangement of successes where the probability of success is known and another arrangement of successes where the probability of success is determined experimentally. In both cases the interest is in the distribution of successes that are relatively infrequent.

3. *Symposium on automatic digital computers.*

High speed electronic computers are being more and more widely used to solve both scientific and industrial problems. The following three papers outlined some of the methods currently used in preparing problems for high speed computation:

(1) *Basic programming*, by Dr. Ruth Goodman,

(2) *Automatic coding*, by Dr. D. H. Shaffer,

(3) *Numerical integration*, by Dr. Morris Ostrofsy,

staff members of the Research Laboratory, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

L. T. MOSTON, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-second annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on May 7, 1955. Two sessions were held at which Professor H. W. Alexander of Earlham College, Chairman of the Section, presided.

There were 60 in attendance, including the following 52 members of the Association:

H. W. Alexander, W. C. Arnold, Juna L. Beal, L. G. Black, A. P. Boblett, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, W. W. Chambers, K. W. Crain, H. E. Crull, M. W. DeJonge, W. E. Edington, P. D. Edwards, C. B. Gass, E. L. Godfrey, S. H. Gould, G. H. Graves, Ralph Hafner, H. H. Hartzler, Ralph Hull, H. L. Hunzeker, M. W. Keller, E. L. Klinger, Florence Long, Gladys B. McColgin, D. M. Mesner, G. T. Miller, Vera T. Morris, J. E. Mueller, R. H. Oehmke, Theresa M. C. Oehmke, Gloria Olive, C. C. Oursler, P. W. Overman, T. P. Palmer, J. C. Polley, D. H. Porter, J. N. Rogers, R. M. Ross, A. R. Schmidt, K. J. Sidebottom, Sister Gertrude Marie, C. P. Sousley, Anna K. Suter, R. O. Virts, M. S. Webster, K. P. Williams, Herbert Wolf, Elizabeth S. Wolf, H. E. Wolfe, G. N. Wollan.

The following officers were elected: Chairman, Mr. R. O. Virts, Central High School, Fort Wayne, Indiana; Vice-Chairman, Professor C. F. Brumfiel, Ball State Teachers College; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

Both sessions were held in the Holcomb Observatory, recently constructed and in use this year for the first time. Professor G. C. McVittee, Director of the Observatory at the University of Illinois, was guest speaker for the hour lecture. The title of the lecture was: *Why should an astronomer study relativity*?

Professor H. E. Crull of Butler University gave a planetarium demonstration following the lecture.

Professor P. D. Edwards, chairman of the Committee on Awards, reported that four Association medals had been awarded for high mathematical achievement in the Indiana Science Talent Search.

The following papers were presented:

1. *Some elementary properties of bonding mappings*, by Professor R. H. Oehmke, Butler University.

In any non-associative algebra A of characteristic not 2, with a subspace S closed under the operation $(x, y) = \frac{1}{2}(xy+yx)$, $U(S)$ denotes the subspace generated by all elements $xy-yx$ for x and y in S. If T is any linear mapping from $U(S)$ into S, a multiplication $x \circ y$ can be defined in S as $x \circ y = \frac{1}{2}(xy+yx)+(xy-yx)T$. Thus a new algebra $B(A, T)$ is defined which is in the same vector space as S and is closed under the product $x \circ y$. This algebra is said to be *bonded* to A by the *bonding mapping* T. The behavior of associative, Jordan, Lie, and power-associative algebras under a bonding mapping was examined. Such tools as ideals, idempotents, derivations, *etc.* used for the study of the structure of algebras were also examined.

2. *The mathematical theory of the Hatchet Planimeter*, by Professor P. D. Edwards, Ball State Teachers College.

The March 1954 issue of *The Professional Geographer* contained a short description of the "Hatchet Planimeter" by R. L. Williams. The mathematical theory was not given. The instrument has been used to a limited extent by cartographers and others since its description by Prytz in 1889. In this paper the mathematical theory is presented and comments made on the degree of accuracy to be expected.

3. *A summary of integral methods*, by Professor T. P. Palmer, Rose Polytechnic Institute.

Integration methods can be summarized under six topics: (1) the integral of $u^n du$, with $n = -1$ as a special case; (2) a collection of eight exact differentials (exponential and trigonometric; eleven, if including hyperbolic functions); (3) integration by parts; (4) substitution (chiefly trigonometric); (5) partial fractions; and (6) trigonometric identities. The last four topics are not really calculus, but provide ways of rearranging so that the first two topics apply. The only differential included which is not familiar from differential calculus is $d \ln (\sec x+\tan x) = \sec x dx$. By these methods, any form whose integral can be expressed in elementary functions can be integrated easily without reference to tables.

4. *Some embedding theorems for incidence matrices*, by Professor D. M. Mesner, Purdue University.

To a given incidence matrix A, matrices B, C, and D are to be adjoined so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an $m \times n$ incidence matrix with equal row totals T and equal column totals U. The required numbers of 0's and 1's in each row of B and column of C, and in the entire block D, are easily computed. It is obviously necessary for the existence of B, C, and D that these numbers be non-negative, and that $mT = nU$. These are shown to be sufficient conditions as well. This generalizes a theorem of Ryser (*Proc. Amer. Math. Soc.*, vol. 2, 1951, 550–552).

5. *Some applications of evenly convex sets*, by Dr. J. R. Blum, Indiana University, introduced by Professor K. P. Williams.

A convex set in finite-dimensional Euclidean space is called evenly convex (W. Fenchel, *A remark on convex sets and polarity*, Comm. du sem. math. de L'un. de Lund, tome supp., 1952) if it is the intersection of a family of open half-spaces. Criteria are given for a convex set to be evenly convex, and for a vector to belong to an evenly convex set. These are applied to obtain an existence criterion for unbiased tests of finite statistical hypotheses.

6. *Mohr space representation of algebraic equations*, by Professor M. O. Peach, University of Notre Dame.

The Mohr circle construction used by engineers to represent the stress tensor is extended to space of higher dimensions, hence to square symmetric matrices of arbitrary rank. It is then generalized to represent non-symmetric matrices. A step by step method for diagonalizing such matrices is interpreted geometrically, both for the case of real and the case of complex characteristic roots. The well known procedure for writing the matrix for which a given algebraic equation is the characteristic equation provides the connecting link whereby any algebraic equation can be given a unique geometrical representation in Mohr space.

7. *The structure of commutative semigroups*, by Professor R. E. MacKenzie, Indiana University, introduced by Professor H. E. Wolfe.

By suitably formulating the basic structure theorems of commutative rings it is possible to carry through their demonstration without the use of the operation of addition. These theorems then become statements about the structure of commutative semigroups. The formulation is such that the theorems on rings may then be obtained by assuming that the semigroup is a ring.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-fourth annual meeting of the Illinois Section of the Mathematical Association of America was held at Monmouth College, Monmouth, Illinois, on May 13 and 14, 1955. Professor Rothwell Stephens, Chairman of the Section, presided at all sessions.

There were 47 in attendance, including the following 37 members:

Beulah M. Armstrong, H. G. Ayre, J. W. Beach, H. R. Beveridge, D. R. Bey, A. H. Black, A. O. Boatman, H. A. Bott, Joseph R. Brown, L. J. Burton, Paul Cramer, Allen Fenstermacher, S. R. Filippone, A. E. Gault, A. E. Hallerberg, M. C. Hartley, F. E. Hohn, M. R. Kenner, E. C. Kiefer, Rose Lariviere, A. O. Lindstrum, Jr., Saunders MacLane, W. G. Madow, W. C. McDaniel, A. W. McGaughey, E. B. Miller, M. G. Moore, C. E. Moulton, T. E. Rine, L. A. Ringenberg, W. C. Ross, Jr., M. Anice Seybold, W. H. Spragens, Jr., Rothwell Stephens, Gabriel Tsiang, L. L. Wimp, Alice K. Wright.

At the business meeting on Friday afternoon the following officers were elected for the coming year: Chairman, Professor H. R. Beveridge, Monmouth College; Vice-Chairman, Professor L. A. Ringenberg, Eastern Illinois State College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. Professor Joseph Stipanowich reported on the work of the "Committee on Contests and Awards" stating that the number of high schools participating increased over that of the preceding year by almost 80%. Professor A. O. Lindstrum, Jr., reported on the work done by the "Committee on the Strengthening of the Teaching of Mathematics" and proposed several resolutions which were adopted.

The following program was presented:

1. *Recent trends in operational calculus*, by Professor E. J. Scott, University of Illinois.

Heaviside's operational calculus and its development in the form of the Laplace transform were briefly discussed and the difference between them indicated. It was shown how certain shortcomings of the Laplace transform theory have been recently overcome by J. G. Mikusinski by returning to a direct interpretation of Heaviside's symbols and giving them a mathematical sense by generalizing the concept of number suitably.

2. *A logical basis for engineering calculus*, by Professor M. E. Munroe, University of Illinois, introduced by the Secretary.

A rigorous justification of the formal manipulation commonly referred to as "engineering calculus" can be based on the following consideration. Definitions of variables and differentials should be independent of the equations in which they appear. Let a variable x be a mapping of something (*e.g.*, times, distances) into numbers, and let its differential dx be a similar mapping. Equations relating variables (including differentials) are conditional equations, and justification of a formal manipulation amounts to checking for consistency the conditional equations that appear in the argument.

3. *A note on the indeterminate forms 0^0 and ∞^0*, by Professor J. W. Beach, Northern Illinos State Teachers College.

This paper presented four theorems concerning the formation of a function which at a point, a, is one of the indeterminate forms 0^0 or ∞^0, but the limit of the function as the variable approaches a is some value other than 0, 1, or ∞.

4. *Fibonacci numbers and Lucas series*, by Professor L. J. Burton, Lake Forest College.

$\tau=\frac{1}{2}(1+\sqrt{5})$ is defined by means of the golden section, and is related to the construction of a regular decagon, the continued fraction $1+1/1\cdots$, and the safe combinations in Wythoff's game. The Fibonacci numbers 1, 1, 2, 3, 5, 8, $\cdots$, are introduced by means of the convergents to τ and the Lucas series 1, 3, 4, 7, 11, $\cdots$, is also defined in terms of τ, and certain simple properties of the series are explained. Finally a proof is outlined by means of congruences in $k(\sqrt{3})$ that if p is an odd prime, then $M_p=2^p-1$ is prime if and only if M_p divides S_{p-1}, where $S_1=4$, $S_n=S_{n-1}^2-2$. With this theorem the SWAC computer in Los Angeles showed in about an hour that $2^{2281}-1$ is a Mersenne prime, the largest specific prime yet known.

5. *What constitutes an effective program of student teaching in secondary mathematics?* by Professor T. E. Rine, Illinois State Normal University.

Improvement of student teaching programs is considered important to mathematics teachers and other educators. This is shown by statements of teachers in service, by the findings of national surveys, and by the work of professional committees.

About 15 years ago the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics gave its support to student teaching in these words:"The most important element in professional training is student practice teaching, carried out under the most competent supervision that can be procured. The commission considers this work so important that it urges that even greater attention be paid to it in the future than in the past." (N.C.T.M., Fifteenth Yearbook, 1940, p. 191).

Since that time very little has been done to improve the program of student teaching in secondary mathematics on a nationwide basis. An effective program of student teaching in secondary

mathematics must take into consideration the needs of prospective teachers in regard to four major aspects, namely: preparation for admission to student teaching, supervision of student teaching, the participation period of student teaching, and evaluation of student teaching. Any educational institution that is concerned with providing outstanding teachers of secondary mathematics should make a careful study and analysis of its program of student teaching in secondary mathematics in each of these major aspects.

6. *The revolution in mathematics instruction*, by Professor Saunders MacLane, University of Chicago.

At the dinner Professor MacLane spoke about the revolution. This revolution springs from the need to introduce new mathematical ideas and methods into the curriculum. Basically, mathematical instruction remains always the same process of the discovery by youth of the beauty of mathematical ideas, but the chances of success are better when youth is exposed to the best mathematical ideas. The process of introducing newer ideas (*e.g.* sets) into undergraduate courses is well under way, and is illustrated by the work of the Illinois Section and the Association at large. The problem of introducing newer mathematical ideas into high schools has just come to the foreground; it offers challenging prospects and difficult questions.

7. *Linear algebra for students of applied mathematics*, by Professor F. E. Hohn, University of Illinois.

This paper outlines a course in linear transformations and matrices given at the University of Illinois for students interested in the applications of the subject. The course is used to introduce the student to a variety of abstract concepts as well as to techniques of computation. The author feels that on the grounds of usefulness and mathematical importance, it might well replace the traditional course in the theory of equations.

8. *A report on a cultural mathematics course*, by Professor W. C. McDaniel, Southern Illinois University.

A course has been given at Southern Illinois University for the past three years to students in the general program who have no particular vocational needs for mathematics. These students have a weak mathematical background on the average. The course stresses understanding rather than techniques. The instructors of the course believe that it is moderately successful and should be continued with further modifications.

9. *A proof of the fundamental theorem of algebra*, by Professor H. E. Vaughan, University of Illinois.

In this proof it is shown that the fundamental theorem of algebra is a simple consequence of the following easily proved generalization of the Brouwer Fixed Point Theorem: If $R>0$ and n is a positive integer, and if f is continuous and $|f(z)| \leqq R^n$ in $|z| \leqq R$, then there exists a number z_0 with $|z_0| \leqq R$ such that $z_0^n = f(z_0)$.

10. *Mathematics for social scientists and its effect on undergraduate teaching*, by Professor W. G. Madow, University of Illinois.

This paper continues and amplifies the discussion of *Mathematics for Social Scientists* (this MONTHLY, vol. 61, 1954, pp. 550–561) and the report of the Social Science Research Council Committee on the Mathematical Training of Social Scientists soon to be published in the Items of the SSRC (copies may be obtained from the present author) with particular emphasis on undergraduate mathematics courses. The main point is that the two years of undergraduate mathematics suitable for social scientists are better for majors in mathematics, statistics, the natural sciences and humanities than the present curriculum; and, with some supplementation, are believed

also to be better for students in engineering and the physical sciences. Experimentation with "non-contractual" groups such as the social scientists and mathematicians is strongly recommended to the end that the necessary curriculum and books may be developed along the lines indicated in the above reports and in this paper.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Teresa in Winona, Minnesota, on May 7, 1955. Sessions were held in the forenoon, at luncheon and in the afternoon. Sister M. Thomas à Kempis, Professor M. D. Donsker, and Sister M. Leontius, Chairman of the Section, presided at the respective sessions.

There were 64 persons in attendance, including the following 37 members of the Association:

N. E. Albrecht, H. M. Anderson, Souren Babikian, K. H. Bracewell, G. U. Brauer, E. J. Camp, C. S. Carlson, H. D. Colson, Brother Louis De La Salle, R. J. Dowling, Mary I. Elwell, J. E. Hafstrom, Fulton Koehler, D. R. Lewis, C. B. Lindquist, W. S. Loud, K. O. May, W. R. McEwen, Reverend Florian Muggli, E. D. Nering, C. R. Perisho, P. C. Rosenbloom, Sister M. Bibiana, Sister M. Helen, Sister M. Joanne, Sister M. Thomas à Kempis, Sister Mary Leontius, Sister Mary Seraphim, F. C. Smith, O. E. Stanaitis, F. J. Taylor, Takashi Terami, Frances E. Walsh, Chih-yi Wang, K. W. Wegner, R. P. Winter, F. L. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Professor E. J. Camp, Macalester College; Secretary, Professor F. C. Smith, College of St. Thomas; Executive Committee, Professor M. D. Donsker, University of Minnesota, Professor H. B. MacDougal, South Dakota State College, Sister Mary Leontius, College of St. Teresa.

By invitation of the Executive Committee, Sister M. Helen of Mount St. Scholastica College delivered an address at the morning session entitled "Some Philosophic Considerations of Mathematics." Abstract of this address follows:

In the first section of this paper the author proposes to answer the following questions: (1) what is mathematics in itself? (2) what comprises its matter and its form? (3) how is mathematics regarded by the layman, the scientist, the mathematician and the philosopher?

The last half of the paper treats of those specific features which marked the field of mathematics in the past and contrasts these with mathematics as we know it today. The concluding paragraphs point out what in mathematics is universal and abiding and what is particular and transient. Special mention is given the role of the imagination in the field of speculative mathematics.

The following short papers were presented:

1. *Seminar in mathematics*, by Professor Frances E. Walsh, College of St. Teresa.

This paper presents a description of a course entitled Seminar in Mathematics offered for seniors majoring in mathematics at the College of St. Teresa. The position of the course in the curriculum is developed, showing the contribution made to the fulfillment of the aim of the college, *i.e.*, the training of apostolic women. The objectives of such a course must conform with the objectives of the institution in which the course is being developed. The ways in which the study of mathematics lends itself to developing intellectual maturity are considered and emphasis is placed on the spirit in which the course is presented.

2. *A problem in probability*, by Professor W. R. McEwen, University of Minnesota, Duluth Branch.

Let P_n be the probability that in throwing a die it will require exactly n throws before each of the numbers one through six has appeared.

$$P_n = \sum_{k=0}^{4} (-1)^k \binom{5}{k} \left(\frac{5-k}{6}\right)^{n-1}; \qquad S_n = \sum_{i=1}^{n} P_i = \sum_{k=0}^{5} (-1)^k \binom{6}{k} \left(\frac{6-k}{6}\right)^n.$$

The most probable value for n is 11. $P_{11} = .08439$; $S_{13} = .5139$; $S_{16} = .6980$; $S_{26} = .9480$; $S_{36} = .9899$.

3. *The rotational component of a certain measure-preserving non-linear transformation in Wiener space*, by Mr. J. E. Hafstrom, University of Minnesota, Duluth Branch.

Consider the Wiener space transformation $Tx(t) = x(t) + q[t; x(t_1), x(1)]$, where $q[t; u_1, u_2] = \int_0^t p[s; u_1, u_2]ds$ and $p[t; u_1, u_2] = \chi(0 \leqq t \leqq t_1)(a_{11}u_1 + a_{12}u_2) + \chi(t_1 < t \leqq 1)(a_{21}u_1 + a_{22}u_2)$. The a_{ij} are now to be determined so that $m_W(\Gamma) = m_W(T\Gamma)$ for every measurable set Γ in C. If the Wiener integral $\int_C F(X)d_W X$ is expressed as a double integral over $C \otimes C$ and is again expressed as a double integral over $C \otimes C$ after first being transformed, comparison of the two double integrals shows the *rotational component* of the transformation to be

$$\sqrt{t_1}\, Y^*\left(\frac{t}{t_1}\right) = \sqrt{t_1}\left[y^*\left(\frac{t}{t_1}\right) + t(a_{11} + a_{12})y^*(1)\right] + t\sqrt{1-t_1}\, a_{12}z^*(1), \qquad 0 \leqq t \leqq t_1$$

and

$$\sqrt{1-t_1}\, Z^*\left(\frac{s-t_1}{1-t_1}\right) = (s-t_1)\sqrt{t_1}\,(a_{21} + a_{22})y^*(1) + \sqrt{1-t_1}\left[z^*\left(\frac{s-t_1}{1-t_1}\right) + (s-t_1)a_{22}z^*(1)\right],$$

$$t_1 \leqq s \leqq 1.$$

Upon dividing out constants and setting $t = t_1$ and $s = 1$, we obtain the system

$$\begin{cases} Y^*(1) = [1 + t_1(a_{11} + a_{12})]y^*(1) + \sqrt{t_1(1-t_1)}\, a_{12}z^*(1) \\ Z^*(1) = \sqrt{t_1(1-t_1)}\,(a_{21} + a_{22})y^*(1) + [1 + (1-t_1)a_{22}]z^*(1), \end{cases}$$

and this is a true rotation in the Euclidean plane if the a_{ij} are properly chosen. The coefficients of $y^*(1)$ and $z^*(1)$ in the first equation are set equal to $\cos\theta$ and $-\sin\theta$, and in the second equation are set equal to $\sin\theta$ and $\cos\theta$. Then, upon solving the four equations, we obtain values for the a_{ij} which make T a measure-preserving transformation.

4. *Pipe cleaners in calculus*, by Professor K. O. May, Carleton College.

One of the major obstacles for students of partial differentiation and multiple integration is their inability to visualize figures in three dimensions. The speaker displayed some models designed to help the students overcome this weakness. The coordinate planes are represented by pieces of hardware cloth, a galvanized wire mesh that can be obtained at any hardware store with squares from one-quarter to one-half inch. Curves in space are represented by pipe cleaner which can be obtained in fifty-foot lengths, in seventeen different colors, and in diameters ranging from one-eighth to three-eighths of an inch. The pipe cleaner is easy to handle, stays in any shape in which it is put, and can be cut with scissors. The hardware cloth is easily cut with snips and can be attached at intervals with short pieces of cleaner. Figures on the blackboard with chalk of the same colors as in the solid model bring the student to a better understanding of plane drawings. A model to illustrate a particular problem in integration can be made in a few minutes.

5. *The Jordan normal form of a variable matrix*, by Professor E. J. Putzer, Macalester College, introduced by Professor H. L. Turrittin.

Let $A(t)$ be an $n \times n$ matrix of analytic functions. An explicit construction is obtained for a non-singular matrix $P(t)$ such that $P^{-1}AP$ is in Jordan normal form. $P(t)$ is analytic except at a

finite number of points which are explicitly located and classified according to the types of singularities $P(t)$ has on them. The construction can be adapted to give an elementary derivation of the Jordan normal form of a constant matrix and the theorems concerning it.

6. *An integral test for convergence of series*, by Professor O. E. Stanaitis, St. Olaf College.

It is shown that $\sum_{x=1}^{\infty} f(x)$ is convergent if the integral $\int_1^{\infty} f(x)dx$ is convergent and if $\int_1^{\infty} f'(x) \sin 2k\pi x dx = O(1/k^p)$ where p is a positive constant.

7. *A certain class of summation methods*, by Dr. G. U. Brauer, University of Minnesota.

A regular matrix A is said to be of type P if it satisfies the following condition: if $\{s_n\}$ and $\{t_n\}$ are two bounded sequences which are evaluated to σ and τ, respectively, by A, then the sequence $\{s_n t_n\}$ is evaluated to $\sigma\tau$ by A. Some properties of matrices of type P are obtained.

8. *A problem on power series in two variables*, by Professor P. C. Rosenbloom, University of Minnesota.

Let $f(x, y)$ be a power series with non-negative coefficients and suppose that the values $f(1, 1) = 1$ and $f(x_1, y_1) = c$ are prescribed, where (x_1, y_1) is a given point in the unit square. What is the greatest possible value for $f(x_2, y_2)$, where (x_2, y_2) is another given point in the unit square? The problem can also be proposed for polynomials of given degree. The extremal function in both cases is a polynomial with at most two non-vanishing coefficients. Some other properties of this extremal polynomial are known, but it has not been determined explicitly.

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The eleventh annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Buffalo, Buffalo, New York, on May 14, 1955. The Chairman of the Section, Professor J. R. F. Kent of Harpur College, presided at the morning session, and the Vice-Chairman, Professor C. E. Rhodes of Alfred University, presided at the afternoon session.

There were 98 persons in attendance, including the following 76 members of the Association:

M. R. Bates, R. L. Beinert, H. F. Bennett, Dorothy L. Bernstein, W. W. Bessell, Jr., A. H. Blessing, H. D. Block, L. L. Brassaw, Jr., A. L. Buchman, F. J. H. Burkett, W. B. Carver, H. S. M. Coxeter, E. J. Downie, A. H. Ettelson, A. G. Fadell, O. J. Farrell, C. W. Foard, A. H. Fox, C. L. Gape, H. M. Gehman, B. H. Gere, D. T. Gianturco, Lillian Gough, N. G. Gunderson, J. R. F. Kent, Violet H. Larney, D. B. Larson, Caroline A. Lester, R. W. MacDowell, R. T. J. Mahoney, J. N. Mangnall, June M. McArtney, C. A. McCarthy, Rudolf Meyer, Irene P. Monahan, Harriet F. Montague, Mabel D. Montgomery, L. J. Montzingo, Jr., D. S. Morse, C. W. Munshower, W. L. Murdock, W. V. Nevins III, C. S. Ogilvy, F. R. Olson, F. D. Parker, J. M. Perry, C. W. Pflaum, Theresa L. Podmele, L. R. Polan, L. D. Potts, J. F. Randolph, A. W. Ransom, C. E. Rhodes, J. B. Rosser, Mary E. Rudin, Joy B. Russek, P. J. Schillo, Edith R. Schneckenburger, L. F. Scholl, Dorothy B. Shaffer, B. B. Sharpe, Sister Florence Marie, Sister Marion, Sister Mary Michael, Ruth W. Stokes, Irwin Stoner, R. F. Tidd, Nura D. Turner, J. P. van Alstyne, J. R. Vanstone, Rev. G. W. Walker, J. F. Wardwell, F. C. Warner, W. G. Warnock, C. B. H. Watson, Frederick R. White.

At the business meeting the following officers were elected: Chairman, Professor C. E. Rhodes, Alfred University; Vice-Chairman, Professor A. J. Coleman, University of Toronto; Secretary, Professor N. G. Gunderson, University of Rochester. The Executive Committee presented a set of proposed By-Laws which it had prepared, and, after discussion and minor amendment, these By-Laws were adopted. The Chairman pointed out that it might be possible to find a new name for the Section more descriptive of its geographical boundaries, and asked that suggestions be sent to the Executive Committee.

The following papers were presented:

1. *The need of mathematics in business research*, by Dr. W. L. Murdock, General Electric Company.

The speaker described the growth of management in business as a profession, emphasizing the present and potential use of mathematical methods by management. He discussed the mathematical topics which should be included in the education of the management trainee. Linear systems are of particular importance.

2. *Teaching computing without a computer*, by Professor J. F. Randolph, University of Rochester.

After considering many of the traditional computing techniques, the class is introduced to automatic computing machinery through a study of a hypothetical computer, the Hypac. Programming is studied, with the class working out various subroutines as problems. Finally, several actual machines are discussed briefly.

3. *Teaching computing with a computer*, by Professor J. B. Rosser, Cornell University.

In teaching computing when a computing machine is available, work must be scheduled so that the class members will not find it necessary to have access to the machine simultaneously. This requires planning and assigning the homework considerably in advance. It also necessitates outside finals, which might well be given out in parts, each part being given out as soon as the class has covered the relevant material. Another point that needs constant attention is the tendency of people who have a computer available to stop doing any thinking and to rely on the computer for everything. It is necessary to keep bringing in problems for which an easy solution can be found, and then upbraiding the class for solving the problem in a routine fashion on the machine instead of finding the easy solution.

4. *Rigor in undergraduate mathematics*, by Professor F. D. Parker, Clarkson College of Technology.

All students of mathematics will benefit from a more rigorous treatment of undergraduate mathematics than is common today. The sciences and professions which use mathematics have all benefited from the rigor which characterizes mathematics. The speaker believes that the present trend is toward the utilitarian aspects of mathematics, and that mathematics, the students, and eventually the sciences and professions will suffer. He also believes that college students are more capable than is generally realized of understanding a more rigorous treatment. The result will be a better understanding of the aims and nature of mathematics and a better use of mathematics.

5. *Problems of a blind student of mathematics*, by Mr. Irving Bentsen, University of Rochester, introduced by Professor N. G. Gunderson.

Most problems encountered by the blind student of higher mathematics may be classified in terms of literacy in mathematics, time consumed in the mechanics of study, unavoidable reliance

on extensive memorization, and perception of geometrical figures. The standard mathematical Braille notation is too cumbersome and limited for efficient use. Recording texts and lectures affords the student an adequate though time consuming means to the literature. The need for instant memorization arises when the classroom teacher states only once an expression written on the blackboard (memory board). In perception of geometrical figures, a distinction is made between those students who have once seen and those who have not. For a student blind from birth problems may arise when a teacher explains a geometrical concept in visual terms. On the other hand, this student is not distracted by representation of figures in perspective, since perspective is an almost meaningless concept to him.

6. *Hyperbolic triangles*, by Professor H. S. M. Coxeter, University of Toronto.

Elegant proofs are described for three well known properties of triangles in the hyperbolic plane. The first is Gauss's proof that the area of a triangle is proportional to its angular defect. The second is a slightly improved version of Liebmann's proof that this area remains finite even when the sides are infinitely long. The third is a projective proof (applying Brianchon's theorem to the absolute conic) for the hyperbolic counterpart of the familiar theorem that the six bisectors of the three angles of a triangle are the six sides of a complete quadrangle.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-third annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Cardinal Stritch College, Milwaukee, Wisconsin, on May 7, 1955. Professor A. E. May, Chairman of the Section, presided.

There were 75 present, including the following 45 members of the Association:

R. H. Bardell, W. R. Brittenham, Leonard Bristow, R. C. Buck, G. L. Bullis, K. L. Clark, E. G. H. Comfort, Miriam E. Connellan, B. F. Dostal, H. P. Evans, J. V. Finch, C. E. Flanagan, Harold Glander, P. C. Hammer, C. B. Hanneken, E. G. Harrell, Fannie Hopkins, R. C. Huffer, Rev. M. L. Jautz, E. R. Johnston, Frances W. Jones, W. E. Lawrence, A. P. Loomer, Morris Marden, A. E. May, J. J. McLaughlin, Karl Menger, Genevieve T. Meyer, A. C. Moeller, O. E. Overn, H. P. Pettit, Berthold Schweizer, Sister M. Elizabeth, Sister Mary Corona, Sister Mary Felice, Sister Mary Petronia, Sister Ruthmary, Abraham Spitzbart, J. C. Spradling, Dorothy J. Stodola, E. W. Swokowski, J. V. Talacko, C. J. Vanderlin, Jr., R. D. Wagner, Louise A. Wolf.

The following officers were elected for the coming year: Chairman, Professor A. C. Moeller, Marquette University; Vice-Chairman, Professor Raphael Wagner, University of Wisconsin; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Program Committee Chairman, Professor Raphael Wagner.

It was decided at the business meeting that the Section would undertake a state-wide mathematics contest for high school students each year. The Executive Committee will appoint a Committee on Contests, which will be responsible for organizing and administering the first of these contests this year.

The following papers were presented:

1. *Antiderivatives, integrals, cumulations*, by Professor Karl Menger, Illinois Institute of Technology. (By invitation.)

(1) $\int \cos x dx = \sin x + c$ is correct when considered as an expression of the idea that any function having the cosine as its derivative differs from the sine by a constant function. As a formula connecting three terms (1) is false. Indeed, (1) connects neither three specific functions as does $\cos^2 x = 1 - \sin^2 x$ for any x, nor function variables, as does the inequality $-f^2(x) \leqq \sin^2 x + c^2$ for any x. Here f may be replaced by any function, *e.g.*, $f(x)$ by $\sin x + 7$, and c by any constant function, say 6. However, in (1) if $\int \cos x dx$ is replaced by $\sin x + 7$, then c must be replaced by 7 only.

Let $\mathbf{D}^{-1}_{(a,b)}f$ denote the *definite antiderivative* of f that assumes the value b at a; $\{\mathbf{D}^{-1}f\}$, the *antiderivative class* consisting of all antiderivatives of f; and $\mathbf{D}^{-1}f$, an *indefinite antiderivative*, *i.e.*, a function variable that may be replaced by any function in $\{\mathbf{D}^{-1}f\}$. Each of these three concepts lends itself to a correct formulation of the idea behind (1).

For details concerning antiderivatives, their connection with integrals, and the application of the latter to the cumulation of one variable quantity with regard to another variable quantity, cf. the author's book, *Calculus. A Modern Approach*, Boston, 1955.

2. *An inequality involving geometric and arithmetic means*, by Professor E. R. Johnston, Wisconsin State College, Whitewater.

It is well known that the geometric mean of a set of positive numbers is bounded above by their arithmetic mean; for example, see Pólya and Szego: *Aufgeben und Lehrsatze aus der Analysis*, Dover, vol. I, p. 50. This paper gives a strictly elementary proof of the following proposition, which is believed to be new, and which expresses a lower bound for the geometric mean in terms of the arithmetic mean.

If $0 < P \leqq X_i \leqq Q$, $i = 1, 2, 3, \cdots, N$, then

$$\left[\prod_{i=1}^{N} X_i\right]^{1/N} \geqq P^{(Q-\xi)/(Q-P)} Q^{(\xi-P)/(Q-P)},$$

where

$$\xi = \frac{\sum_{i=1}^{N} X_i}{N}.$$

3. *The use of high speed computers in solving problems*, by Professor P. C. Hammer, University of Wisconsin.

The high speed computing machines make it possible to utilize more fully the mathematical theories of the past in applications of various kinds. It is now possible to solve feasibly rather large linear equation systems, to invert matrices, to approximate numerically the solutions of ordinary differential equation systems and of certain partial differential equations. In part the high speed computing machine has made mathematics more useful and in part it has revealed a need for much sharper theories in some directions. Except for linear systems, no effective theory of roundoff errors has been given. Despite the advertising, it is very simple to pose problems not feasible for any existing computing machine.

4. *Mathematics in the social sciences*, by Mr. W. A. Golomski, Marquette University.

5. Panel Discussion: *Problems of the colleges in regard to the high school mathematics preparation of their students.*

Leaders: Professor J. J. McLaughlin, Wisconsin State College, River Falls; Professor E. G. Harrell, Wisconsin State College, Platteville; Professor H. P. Evans, University of Wisconsin.

Professor McLaughlin presented a brief report of the findings of the college section of the Wis-

consin State-wide Mathematics Curriculum Committee in their study of the mathematics curriculum in the secondary school for college preparatory students.

Professor Evans pointed out that the placement of entering freshmen at the University of Wisconsin in the mathematics sequence, based on the students' records and the result of a machine scored placement test, is considerably lower than it would be if the placement were based on the high school record of mathematical preparation only.

Professor Harrell called attention to the following problems: (a) since state colleges must accept any one presenting a high school diploma, the mathematics background of students varies greatly requiring a diversified college mathematics program; (b) the lack in the student of a will to learn something; (c) the difficulty with which the student changes from supervised to independent study habits; (d) lack of uniformity of curricula and pre-requisites.

The consensus of opinion seemed to be that the colleges should provide two or more track courses but not offer high school mathematics courses, do less coddling and provide more incentive.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The ninth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of British Columbia, Vancouver, British Columbia, on June 17, 1955, in conjunction with the five hundred fifteenth meeting of the American Mathematical Society. Professor Ivan Niven, Chairman of the Section, presided at the meetings.

There were 71 persons in attendance, including the following 43 members of the Association:

C. B. Allendoerfer, J. P. Ballantine, R. A. Beaumont, J. L. Brenner, L. G. Butler, Harold Chatland, Paul Civin, P. A. Clement, D. F. Coulter, Jr., D. B. Dekker, Douglas Derry, W. H. Gage, H. M. Gelder, K. S. Ghent, S. G. Hacker, Mary E. Haller, Edwin Hewitt, E. Y. Hill, T. E. Hull, Burrowes Hunt, H. H. Irwin, D. H. Jones, E. S. Keeping, J. M. Kingston, V. L. Klee, Jr., M. S. Knebelman, A. E. Livingstone, A. T. Lonseth, J. E. Maxfield, Granville McCormick, L. V. Mead, A. F. Moursund, B. N. Moyls, D. C. Murdoch, Ivan Niven, T. G. Ostrom, C. A. Pursel, J. L. Simpson, P. O. Steen, C. A. Swanson, J. R. Vatnsdal, R. T. Wallace, R. M. Winger.

Following a joint dinner meeting with the American Mathematical Society a business meeting was held in the evening in Brock Hall. The following officers were elected: Chairman, Professor D. C. Murdoch, University of British Columbia; Vice-Chairman, Professor S. G. Hacker, Washington State College; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon. Reports of the High School Mathematics Contests in British Columbia, Washington and Oregon were received. A program committee consisting of Professor Arvid Lonseth, Professor Ostrom, Professor Murdoch and Professor Beaumont was appointed. Since the national meeting of the Association is to be held in Seattle in August, 1956, it was decided that no Pacific Northwest Section meeting will be scheduled for 1956.

The afternoon session consisted of two fifteen minute papers, an invited hour address and a symposium on freshman mathematics.

1. ***An interpretation of differentials***, by Professor C. B. Allendoerfer, University of Washington.

Let $f(x)$ be a differentiable function of the n-variables x^i, and let $\xi^i(x)$ be an arbitrary contravariant vector field. Then the differential of $f(x)$ is defined to be $df(x) = (\partial f(x)/\partial x^i)\xi^i(x)$. This definition is invariant under coordinate transformations. The second differential is given by $d_{\eta\xi}f = (\partial^2 f/\partial x^i \partial x^j)\xi^i\eta^i + (\partial f/\partial x^j)(\partial \xi^j/\partial x^i)\eta^i$; this also is an invariant definition. When ξ^i and η^i are both constant in the x-coordinate system $d_{\eta\xi}f = d_{\xi\eta}f = (\partial^2 f/\partial x^i \partial x^j)\xi^i\eta^i$; this is not an invariant statement. If the first differentials of n independent functions are given, the vector $\xi^i(x)$ can be obtained. Hence vectors and differentials are equivalent ideas. The modern definition of a tangent vector introduced by Chevalley is pertinent here.

2. *S-numbers, an extension of Fermat numbers* (preliminary report), by J. E. and Margaret W. Maxfield, Naval Ordnance Test Station, China Lake, California, presented by Mrs. Maxfield.

Let s be a positive integer. An integer N will be called an s-number if N is greater than s and $a^{N-s} \equiv 1 \pmod{N}$ for every integer a prime to N. It is shown that an infinitude of s-numbers exist for each s.

The problem of s-numbers is linked to investigation of $\lambda(N-s)$ and to the relationship between λ and ϕ. It is shown that the highest power of an odd prime that can divide an s-number is one more than the highest power that divides s. It is shown that if an s-number is a product of two primes p, q, then $p+q \leqq s+1$. Conditions are found for s-numbers to have the form $p^k q^m$ when $s = p^i q^j$ and the form $2^k p^m$ when $s = 2^i p^j$.

3. *Statistical decisions*, by Professor E. S. Keeping, University of Alberta. (By invitation.)

Wald's decision theory ranks, along with Fisher's small-sample theory and the Neyman-Pearson theory of estimation, as one of the great advances in mathematical statistics. The relation of decision theory with the theory of games is discussed, the statistician being supposed engaged in a game with Nature, in which either side may employ pure or mixed strategies. Wald's minimax principle is compared with other decision criteria, including those of Laplace, Savage and Hurwicz.

Examples are given in which the minimax principle leads to a different estimate from that obtained by the customary procedures. A problem in sequential analysis is treated from the point of view of decision theory.

4. *Symposium on freshman mathematics*. Moderator: Professor M. S. Knebelman, Washington State College.

Speakers:

(1) Professor S. G. Hacker, Washington State College.

Two freshman mathematics populations were recognized: (1) students who plan to take only one year of mathematics and (2) students who will take non-elementary mathematics. A proposal was made that all member institutions of this Section consider giving a uniform test to determine capabilities of students for admittance to first year "college" mathematics. The training of all students should be mathematically sound with emphasis on (1) mathematical concepts of fundamental significance, (2) methods of mathematical proof with rigor commensurate with the student's capabilities and mathematical maturity and (3), less importantly, technique. No explicitly topical program of instruction was given, but illustrative ideas and proofs were cited.

(2) Professor Edwin Hewitt, University of Washington.

The following remarks are based on the writer's experience in teaching a liberal arts course in mathematics to very heterogeneous classes at the University of Washington. It was found that the calculus of propositions and simple applications to verbal problems are easily learned. The logic of "$\forall$" and "$\exists$" is grasped by some but not all. The axiomatic theory of fields, suitably disguised as a rigorous discussion of elementary algebra, excites great interest. Finite fields, however, offer serious difficulties to many. Deductive systems, with a rigorously defined language and axioms,

are definitely accessible. Curiously, the theory of finite groups seems hard, although in a class of 25 there are usually 2 or 3 students who can compute all groups of order 6, thus getting an automatic "A."

(3) Professor A. F. Moursund, University of Oregon.

The freshman mathematics course for liberal arts students should be reasonably satisfactory for: the large group of students who will take no further work in mathematics; those students, principally physical science majors, who will take calculus; an increasing number of students, principally social and biological science majors, who will take some work in statistics. The author proposes a general course, more of the traditional type than that included in the recent Allendoerfer and Oakley text. He advocates that the usual treatments of trigonometry and analytic geometry be greatly curtailed; that certain traditional algebraic topics be omitted and others simplified; that some work in the calculus and in elementary statistics be included; and that basic mathematical ideas, the nature of mathematical proof, and understanding be emphasized along with the attainment of fair manipulative skill.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA, South Dakota State College, Brookings, October 15, 1955.

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April, 1956.

OKLAHOMA, Oklahoma City University, October 28, 1955.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

ROCKY MOUNTAIN

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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